浦部 治一郎

論文内容の要旨

報告番号	乙第　　号	氏名	浦部治一郎
論文調査担当者	主査 溝畑 茂 山口昌哉 楠 幸男		

（論文題目）

Hamada's Theorem for a certain type of the operators with double characteristics.

（2重特性根をもつある種の作用素に対する浜田の定理）

（論文内容の要旨）

原点の近傍で定義されたm階の解析的係数をもつ偏微分方程式の非特性初期値問題を考える。

$$L(x,D)u(x)=0$$

$$D_0^h u(o,x') = \frac{w_h(x_2,\cdots,x_n)}{x_1^{\ell}} \quad (0\leqq h\leqq m-1)$$

ここで $x=(x_0,x')=(x_0,x_1,x_2,\cdots,x_n)$. Cauchy-Kowalewskiの定理と異るのは、初期データが初期面 $x_0=0$ 上の $x_1=0$ で極をもつことである。この研究の主な目的は、$u(x)$ の特異性（singularity）が現れる場所と種類を明確に示すような $u(x)$ の表示を求めることである。理論の性格上複素空間で考える。浜田はLの主要部が単純特性である場合、つづいて多重度が一定で高々2である場合について論じ、その特異性を明らかにした。申請者は参考論文1.2において Tricomi型の偏微分方程式

を取り扱った。Tricomi 型とは、

$$\partial_t^2 - t\partial_x^2 + (\text{first order})$$

であり、参考論文2では、

$$L = P(x,D)^2 - x_0 Q(x,D) + R(x,D)$$

の形の方程式を扱っている。P.Qはそれぞれ斉次m,2m次の作用素、Rは(2m−1)次以下である。申請者は、このとき多重度が一定である場合と異る様相を示すことを明らかにした。相関数$\varphi^{\pm}$は、

$P(x,\nabla\varphi)^2 - x_0 Q(x,\nabla\varphi) = 0$をみたし、$\varphi^{\pm}(o,x^1) = x_1$, $D_0\varphi^{\pm}(o,x^1) = \lambda$をみたすものとして定義されるが、さらに、$\varphi^{\pm} = \rho \pm \frac{2}{3}\theta^{\frac{3}{2}}$という2つの解析的関数($\rho,\theta$)で表示される。$\lambda$は$P_m(x,\lambda,1,0,\cdots 0) = 0$の根であり、m個の相異る$\lambda_\beta$をもつとし、$Q(x,\lambda_\beta,1,\cdots 0) \neq 0$とする。このとき、作用素$\partial_\theta^2 - \theta\partial_\rho^2$の適当な独立解$X(\theta,\rho)$, $Y(\theta,\rho)$をとり、$X(\theta(x),\rho(x))$, $Y(\theta(x),\rho(x))$を展開の基底として採用している。

主論文は、上の後をうけて

$$L = P(x,D)^2 - x_0^2 Q(x,D) + R(x,D)$$

の場合を論じている。P,Qについては前と同一であるが、さらに subprineipal symbol L_Sが特性面上で定数となる仮定がつけ加えられている。このとき$u(x)$は、特性面$\{x;\varphi_\beta^{\pm}(x) = 0\}$, $1 \leq \beta \leq m$を除いた集合の上の普遍被覆空間上で1価正則関数として一意的に定まることが示されている。具体的には次の形となる。$F(\alpha,\beta,\gamma;z)$をガウスの超幾何関数とし、

$$X_{\alpha,\beta}^{(q)}(\theta,\rho) = \frac{1}{q!}\partial_\alpha\partial_c^q\left[F\left(-\alpha-q,\frac{1+c}{4}\cdot\frac{1}{2};1-\frac{\varphi+}{\varphi-}\right)\frac{(\varphi^-)^{\alpha+q}}{\Gamma(\alpha+1)}\right]\Bigg|_{c=c_\beta},$$

$$Y_{\alpha,\beta}^{(q)}(\theta,\rho) = \frac{1}{q!}\partial_\alpha\partial_c^q\left[F\left(-\alpha-q,\frac{3+c}{4},\frac{3}{2};1-\frac{\varphi+}{\varphi-}\right)\frac{(\varphi-)^{\alpha+q}}{\Gamma(\alpha+1)}\right]\Bigg|_{c=c_\beta}$$

を定義する。ここで C_β は特性面上の subprincipal symbol の値である。

$$u(x)=\sum_\beta\sum_\alpha\sum_q \Big\{ u^{(q)}_{\alpha,\beta}(x)\, Y^{(q)}_{\alpha-1,\beta}(\theta_\beta,\rho_\beta)+g^{(q)}_{\alpha,\beta}(x)\,\partial_\theta X^{(q)}_{\alpha,\beta}(\theta_\beta,\rho_\beta)$$

$$+v^{(q)}_{\alpha,\beta}(x)\, Y^{(q)}_{\alpha-1,\beta}(\theta_\beta,\rho_\beta)+h^{(q)}_{\alpha,\beta}(x)\,\partial_\theta\, Y^{(q)}_{\alpha,\beta}(\theta_\beta,\rho_\beta)\Big\}$$

ここで係数はすべて正則である。この結果、特性面上で u(x) は、

$$(\varphi^{\pm}_{\beta})^{\frac{1}{4}(1\mp c_\beta)+i+j}\,(\log\,\varphi^{\pm}_{\beta})^{i},\quad (\varphi^{\pm}_{\beta})^{\frac{1}{4}(3\mp c_\beta)+i+j}\,(\log\,\varphi^{\pm}_{\beta})^{i}$$

$(i=0,1,2,\cdots;\ \ j=-\ell,-\ell+1,\cdots)$ の重ね合わせの特異性をもつことが示されている。

主論文

Hamada's theorem for a certain type of the operators with double characteristics.

by

Jiichiroh Ūrabe

Introduction

We consider non-characteristic Cauchy problem with meromorphic data for a linear partial differential equation with holomorphic coefficients in the complex domain.

This problem, for the operator with constant multiple characteristics, has been investigated by Y. Hamada, J. Leray and C. Wagshal [2] and others in bibliography of [2] . This problem, for the operator with involutive characteristics, has been investigated by Y. Hamada and G. Nakamura [3] , [7] and D. Shiltz , J. Vaillant et C. Wagshal [9] and T. Kobayashi [6] . The author treated this problem for a certain class of operators containing Tricomi operator in [10] , [11] .

We shall treat the most general case but it will be very difficult to solve the problem. In this paper, we treat the limited class of operators originated from $P_c = \partial_t^2 - t^2\partial_x^2 - c\partial_x$ (c is a constant). Our method to solve this problem, is to construct the formal solution as the series of the auxiliary functions with the holomorphic coefficients . Those auxiliary functions , described precisely in the appendix, are composed mainly of hypergeometric functions , whose monodromy theory makes the ramification of the solution around the characteristic surfaces clear, and the convergence of the formal solution valid

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§ 1. Assumptions and results.

Let Ω be a neighbourhood of the origin of C^{n+1}, with the coordinates $x=(x_0,x_1,\cdots,x_n)$. By $L^k(\Omega)$, we mean the set of all linear partial differential operators of order k of which coefficients are holomorphic in Ω. We shall be studying a linear partial differential operator $L(x,D) \in L^{2m}(\Omega)$ with the principal symbol $\overset{\circ}{L}(x,\xi)$ of the following form:

$$\overset{\circ}{L}(x,\xi) = P(x,\xi)^2 - x_0^2 Q(x,\xi)$$

,where $\xi=(\xi_0,\xi_1,\cdots,\xi_n)$ and $D=(D_0,D_1,\cdots,D_n)$, $D_i=\frac{\partial}{\partial x_i}$.

We shall impose on $P(x,\xi)$ and $Q(x,\xi)$ the following conditions:

Assumptions

(P) (i) $P(x,\xi)$ is a homogeneous polynomial in ξ of degree m.

(ii) $P(x,1,0,\cdots,0)\neq 0$.

(iii) The equation $P(0,\xi_0,1,0,\cdots.0)=0$ has mutually distinct m roots λ_β $(\beta=1,\cdots,m)$.

(Q) (i) $Q(x,\xi)$ is a homogeneous polynomial in ξ of degree 2m.

(ii) $Q(0,\lambda_\beta,1,0,\cdots,0)\neq 0$, $(\beta=1,\cdots,m)$.

Then there exist 2m characteristic surfaces $K_\beta^{\pm}$ $(\beta=1,\cdots,m)$ issuing from (n-1)-plane $x_0= x_1= 0$. $K_\beta^{\pm}$ are defined by the equations $\varphi_\beta^{\pm}(x)=0$. Here $\varphi_\beta^{\pm}(x)$ are the solutions of the following eikonal equation:

$$\begin{cases} \overset{\circ}{L}(x,\ \varphi_{\beta\ x}^{\pm}(x))=0 \\ \varphi_\beta^{\pm}(0,x')=x_1\ , \text{ and } \varphi_{\beta\ x_0}^{\pm}(0)=\lambda_\beta\ , \quad (x'=(x_1,\cdots,x_n)). \end{cases}$$

(In § 4, we shall study phase functions $\varphi_\beta^{\pm}(x)$ precisely.)

We write $K=\bigcup_{\beta=1}^{m} K_\beta^{\pm}$. And according to assumptions (P)(i), (ii)and (iii) , $P(x,\xi)$ is decomposed in the following,

$$P(x,\xi) = P(x,1,0,\cdots,0)\prod_{\beta=1}^{m}(\xi_0-\lambda_\beta(x,\xi')),$$

where $\lambda_\beta(x,\xi')$ are holomorphic in $(x,\xi')=(x,\xi_1,\cdots,\xi_n)$ near $(x,\xi')=(0.1.0;\cdots,0)$ and mutually distinct and satisfy $\lambda_\beta(0,1.0,\cdots,0)=\lambda_\beta$. We put $\lambda_\beta(x')=\lambda_\beta(0,x',1,0,\cdots,0)$ and $\rho_\beta=(0,x',\lambda_\beta(x'),1,0,\cdots.0)$.

Furthermore, we shall impose on the symbol $L_s(x,\xi)$ of

$L_s(x,D)$ which is the homogeneous part of order $2m-1$ of $L(x,D)$, the following condition:

Assumption (L_s) (i) $\left\{P^{(0,0)}(e_\beta)\sqrt{Q(e_\beta)}\right\}^{-1}\Big[-2L_s(e_\beta)P^{(0)}(e_\beta)+P^{(0,0)}(e_\beta)P_{x_0}(e_\beta)+\sum_{j=1}^{n}\lambda_{\beta x_j}(x')\left\{P^{(0,0)}(e_\beta)P^{(j)}(e_\beta)-2P^{(0)}(e_\beta)P^{(0,j)}(e_\beta)\right\}\Big]=c_\beta$ (c_β is a constant depending only on β),

where $P^{(j)}(x,\xi)=D_{\xi_j}P(x,\xi)$ and $P^{(i,j)}(x,\xi)=D_{\xi_i}D_{\xi_j}P(x,\xi)$

Note: We give below three simple examples of $L(x,D)$ which satisfy Assumptions (P),(Q) and (L_s).

(Ex.0) $L = D_0^2 - x_0^2D_1^2 - cD_1$ (c is a constant.)

We call this operator P_c henceforth.

(Ex.1) $L = D_0^2 - x_0^2D_1^2 - b(x)D_1$,

where $b(0,x')=c$ (c is a constant.) ($\Leftarrow (L_s)$(i)).

(Ex.2) $L = D_0^2 - x_0^2Q(x,D') + R(x,D)$,

where $Q(x,D')=\sum_{i,j=1}^{n}q_{i,j}(x)D_iD_j$ and $q_{11}(0)\neq 0$ ($\Leftarrow$ (Q)),

$$R(x,D) = \sum_{i=0}^{n}r_i(x)D_i + s(x) \text{ and } \frac{-r_1(0,x')}{\sqrt{q_{11}(0,x')}} = c$$

(c is a constant.) ($\Leftarrow (L_s)$(i)) .

Now, we consider the non-characteristic Cauchy problem with singular data

$$(1.1)\qquad \begin{cases} L(x,D)u(x)=0 \\ D_0^{\,h}u(0,x')=W_h(x') \qquad (h=0,\cdots,2m-1) \end{cases}$$

where at least one of $W_h(x')$ has poles along $x_0=x_1=0$.

To study this Cauchy problem, and its solution, namely its singularities, we need the auxiliary functions $X_\alpha^{(j)}$ and $Y_\alpha^{(j)}$. First we introduce the so-called wave forms $f_\alpha(\rho)$ and $k_\alpha(\rho)$:

$$f_\alpha(\rho)=\frac{\rho^\alpha}{\Gamma(\alpha+1)} \quad ,$$

and

$$k_\alpha(\rho)=\frac{\partial}{\partial\alpha}f_\alpha(\rho)=\begin{cases} \dfrac{\rho^\alpha}{\Gamma(\alpha+1)}(\log\rho-\psi(\alpha+1)) \\ \text{especially } |\alpha+1|!\,(-1)^{\alpha-1}\rho^\alpha \quad \text{for } \alpha=-1,-2,\cdots \end{cases}$$

where $\psi(\alpha)$ is di-Γ function, namely $\dfrac{\frac{d}{d\alpha}\Gamma(\alpha)}{\Gamma(\alpha)}$ and α is a complex parameter.

Next we introduce the fundamental auxiliary functions $U_\alpha(\theta,\rho,C,)$ and $V_\alpha(\theta,\rho,C)$ as the solutions of the Cauchy problems for the operator $P_c=\partial_\theta^2-\theta^2\partial_\rho^2-c\partial_\rho$ (c is a complex paramenter) respectively: $P_c\, U_\alpha(\theta,\rho,c) = 0$

with initial data $\begin{cases} U\ (0,\rho.c)=f_\alpha(\rho) \\ U_{\alpha\theta}(0,\rho,c)=0 \end{cases}$

and $P_c\, V_\alpha\ (\theta,\rho,c) = 0$

with initial data $\begin{cases} V_\alpha(0,\rho,c)=0 \\ V_{\alpha\theta}(0,\rho,c)=f_\alpha(\rho) \end{cases}$

We remark that the following explicit representation of U_α and V_α are known:

$$\begin{cases} U_\alpha(\theta.\rho,c)=\dfrac{(\varphi^-)^\alpha}{\Gamma(\alpha+1)}\ F\left(-\alpha,\dfrac{1+c}{4},\dfrac{1}{2},z\right) \\ V_\alpha(\theta,\rho,c)=\dfrac{(\varphi^-)^\alpha}{\Gamma(\alpha+1)}\ \theta F\left(-\alpha,\dfrac{3+c}{4},\dfrac{3}{2},z\right) \end{cases}$$

where $\varphi^{\pm}=\rho\pm\frac{1}{2}\theta^2$ and $z=1-\dfrac{\varphi^+}{\varphi^-}$.

We define the auxiliary functions $X_\alpha(\theta,\rho,c)$ and $Y_\alpha(\theta,\rho,c)$:

$$\begin{cases} X_\alpha(\theta.\rho,c) = \partial_\alpha U_\alpha(\theta,\rho,c), \\ Y_\alpha(\theta,\rho,c) = \partial_\alpha V_\alpha(\theta,\rho,c). \end{cases}$$

Therefore X_α satisfies the Cauchy problem $P_c X_\alpha=0$ with the initial data $X_\alpha(0,\rho,c)=k_\alpha(\rho)$ and $X_{\alpha\theta}(0,\rho,c)=0$.

And Y_α satisfies the Cauchy problem $P_c Y_\alpha=0$ with the initial data $Y_\alpha(0,\rho,c)=0$ and $Y_{\alpha\theta}(0,\rho,c)=k_\alpha(\rho)$.

We introduce the auxiliary functions $U^{(q)}_\alpha(\theta,\rho,c)$ and $V^{(q)}_\alpha(\theta,\rho,c)$ as the solutions of the Cauchy problem inductively $(q=0,1,2,\cdots)$:

First we set $U^{(0)}_\alpha(\theta,\rho,c)=U_\alpha(\theta,\rho,c)$ and $V^{(0)}_\alpha(\theta,\rho,c)=V_\alpha(\theta,\rho,c)$.

$$P_c U^{(q)}_\alpha(\theta,\rho,c)=U^{(q-1)}_\alpha(\theta,\rho,c) \qquad (\text{for } q\geqq 1)$$

with null initial data $\begin{cases} U^{(q)}_\alpha(0,\theta,c)=0 \\ U^{(q)}_{\alpha\,\theta}(0,\theta,c)=0. \end{cases}$

$$P_c V^{(q)}_\alpha(\theta,\rho,c)=V^{(q-1)}_\alpha(\theta,\rho,c) \qquad (\text{for } q\geqq 1)$$

with null initial data $\begin{cases} V^{(q)}_\alpha(0,\rho,c)=0 \\ V^{(q)}_{\alpha\,\theta}(0,\rho,c)=0. \end{cases}$

Finally we reach the definition of $X^{(q)}_\alpha(\theta,\rho,c)$ and $Y^{(q)}_\alpha(\theta,\rho,c)$ which play important role in this paper.

For $q=0$, we set $X^{(0)}_\alpha(\theta,\rho,c)=X_\alpha(\theta,\rho,c)$ and $Y^{(0)}_\alpha(\theta,\rho,c)=Y_\alpha(\theta,\rho,c)$. For $q\geqq 1$, we define $X^{(q)}_\alpha$ and $Y^{(q)}_\alpha$ as the solutions of the Cauchy problems:

$$P_c X_\alpha^{(q)}(\theta,\rho,c)=X_\alpha^{(q-1)}(\theta,\rho,c) \qquad (\text{for } q\geq 1\ )$$

with null initial data $\begin{cases} X_\alpha^{(q)}(0,\rho,c)=0 \\ X_{\alpha\,\theta}^{(q)}(0,\rho,c)=0. \end{cases}$

$$P_c Y_\alpha^{(q)}(\theta,\rho,c)=Y_\alpha^{(q-1)}(\theta,\rho,c) \qquad (\text{for } q\geq 1)$$

with null initial data $\begin{cases} Y_\alpha^{(q)}(0,\rho,c)=0 \\ Y_{\alpha\,\theta}^{(q)}(0,\rho,c)=0. \end{cases}$

Therefore $X_\alpha^{(q)}(\theta,\rho,c)=\partial_\alpha U_\alpha^{(q)}(\theta,\rho,c)$ and $Y_\alpha^{(q)}(\theta,\rho,c)=\partial_\alpha V_\alpha^{(q)}(\theta,\rho,c)$ hold.

These auxiliary functions f_α, k_α, $U_\alpha^{(q)}$, $V_\alpha^{(q)}$, $X_\alpha^{(q)}$, $Y_\alpha^{(q)}$ satisfy the relations $\frac{d}{d\rho}f=f_{\alpha-1}$, $\frac{d}{d\rho}k_\alpha=k_{\alpha-1}$, $\partial_\rho U_\alpha^{(q)}=U_{\alpha-1}^{(q)}$, $\partial_\rho V_\alpha^{(q)}=V_{\alpha-1}^{(q)}$, $\partial_\rho X_\alpha^{(q)}=X_{\alpha-1}^{(q)}$, $\partial_\rho Y_\alpha^{(q)}=Y_{\alpha-1}^{(q)}$,respectively.

To describe the multi-valued functions $U_\alpha^{(q)}$, $V_\alpha^{(q)}$, $X_\alpha^{(q)}$ and $Y_\alpha^{(q)}$ precisely, we need the following lemma.

LEMMA 1.1. We have the following explicit representations of these multi-valued functions

$$U_\alpha^{(q)}(\theta,\rho,c)=\frac{1}{q!}\partial_c^q U_{\alpha+q}(\theta,\rho,c),$$

$$V_\alpha^{(q)}(\theta,\rho,c)=\frac{1}{q!}\partial_c^q V_{\alpha+q}(\theta,\rho,c),$$

$$X_\alpha^{(q)}(\theta,\rho,c) = \frac{1}{q!}\partial_c^q X_{\alpha+q}(\theta,\rho,c) = \frac{1}{q!}\partial_\alpha \partial_c^q U_{\alpha+q}(\theta,\rho,c),$$

$$Y_\alpha^{(q)}(\theta,\rho,c) = \frac{1}{q!}\partial_c^q Y_{\alpha+q}(\theta,\rho,c) = \frac{1}{q!}\partial_\alpha \partial_c^q V_{\alpha+q}(\theta,\rho,c).$$

Therefore
$$U_\alpha^{(q)} = \frac{1}{q!}\partial_c^q \left[\frac{(\varphi^-)^{\alpha+q}}{\Gamma(\alpha+1)} F\left(-\alpha-q, \frac{1+c}{4}, \frac{1}{2}, z\right)\right],$$

$$V_\alpha^{(q)} = \frac{1}{q!}\partial_c^q \left[\frac{(\varphi^-)^{\alpha+q}}{\Gamma(\alpha+1)} \theta\, F\left(-\alpha-q, \frac{3+c}{4}, \frac{3}{2}, z\right)\right],$$

$$X_\alpha^{(q)} = \frac{1}{q!}\partial_c^q \partial_\alpha \left[\frac{(\varphi^-)^{\alpha+q}}{\Gamma(\alpha+1)} F\left(-\alpha-q, \frac{1+c}{4}, \frac{1}{2}, z\right)\right].$$

$$Y_\alpha^{(q)} = \frac{1}{q!}\partial_c^q \partial_\alpha \left[\frac{(\varphi^-)^{\alpha+q}}{\Gamma(\alpha+1)} \theta\, F\left(-\alpha-q, \frac{3+c}{4}, \frac{3}{2}, z\right)\right].$$

Now we describe our theorem. We put $X_\alpha^{(q)}(\theta,\rho,c_\beta) = X_{\alpha,\beta}^{(q)}(\theta,\rho)$ and $Y_\alpha^{(q)}(\theta,\rho,c_\beta) = Y_{\alpha,\beta}^{(q)}(\theta,\rho)$.

THEOREM 1.1. Under Assumptions (P), (Q)& (L_s), for $r > 0$ sufficiently small, the Cauchy problem (1.1) has a unique holomorphic solution on the universal covering space over $D_r \setminus K$,where $D_r = \{x \in \Omega ; |\varphi_\beta^{\pm}(x)| < r\}$. More precisely speaking, the solution u(x) is given in the following form:

$$u(x) = \sum_{\beta=1}^{m} \sum_{\alpha=-1}^{+\infty} \quad {}_{+1} \sum_{q=0}^{+\infty} \Big\{ u_{\alpha,\beta}^{(q)}(x) X_{\alpha-1,\beta}^{(q)}(\theta_\beta(x), \rho_\beta(x)) +$$
$$g_{\alpha,\beta}^{(q)}(x)\partial_\theta X_{\alpha,\beta}^{(q)}(\theta_\beta(x),\rho_\beta(x)) + v_{\alpha,\beta}^{(q)}(x) Y_{\alpha-1,\beta}^{(q)}(\theta_\beta(x),\rho_\beta(x))$$
$$+ h_{\alpha,\beta}^{(q)}(x)\partial_\rho Y_{\alpha,\beta}^{(q)}(\theta_\beta(x),\rho_\beta(x))\Big\}$$

/

,where l is the highest order of poles of the initial data and $u^{(q)}_{\alpha,\beta}(x)$, $g^{(q)}_{\alpha,\beta}(x)$, $v^{(q)}_{\alpha,\beta}(x)$, $h^{(q)}_{\alpha,\beta}(x)$, $\theta_\beta(x)$ and $\rho_\beta(x)$ are holomorphic in D_r, (as for $\theta_\beta(x)$ and $\rho_\beta(x)$ such that $\varphi^{\pm}_\beta(x) = \rho_\beta(x) \pm \frac{1}{2}\left[\theta_\beta(x)\right]^2$, see § 4) .

For the proof of this theorem, we construct the formal solution of the Cauchy problem (1,1) in the above form, and then confirm the convergence of the formal solution . This theorem shows that the singularities of the solution u(x) are reduced to the singularities of the auxiliary functions $X^{(q)}_{\alpha,\beta}$ and $Y^{(q)}_{\alpha,\beta}$ which are to be studied in Appendix in detail. To construct the formal solution, first we are to prepare some caluculations and some properties of operator L(x,D) and the auxiliary functions ,with which we start in the next section.

§ 2. Preliminary caluculation

To construct the formal solution of the Cauchy problem (1.1), we substitute series of the formal solution in L(x,D)u(x) and caluculate it. To do so, we need to represent $\partial_\rho^i \partial_\theta^j X_\alpha^{(l)}$ and $\partial_\rho^i \partial_\theta^j Y_\alpha^{(l)}$ in terms of $\partial_\rho^k X_\alpha^{(l-q)}$ & $\partial_\rho^{k-1}\partial_\theta X_\alpha^{(l-q)}$ and $\partial_\rho^k Y_\alpha^{(l-q)}$, & $\partial_\rho^{k-1}\partial_\theta Y_\alpha^{(l-q)}$ respectively. So we employ the following formula:

$$
\text{(F1)}\begin{cases}
\partial_\theta^{2r} U_\alpha^{(j)} = \sum_{k=0}^{r} \sum_{s=2k}^{r+k} c_{s,k}^{2r} U_{\alpha-2r+s}^{(j-k)} \\
\qquad + \sum_{k=0}^{r-1} \sum_{s=2k+1}^{r+k} d_{s,k}^{2r} \partial_\theta U_{\alpha-2r+s+1}^{(j-k)} \\
\partial_\theta^{2r+1} U_\alpha^{(j)} = \sum_{k=0}^{r} \sum_{s=2k}^{r+k} c_{s,k}^{2r+1} U_{\alpha-2r+s}^{(j-k)} \\
\qquad + \sum_{k=0}^{r} \sum_{s=2k}^{r+k} d_{s,k}^{2r+1} \partial_\theta U_{\alpha-2r+s}^{(j-k)}
\end{cases}
$$

where $c_{s,k}^{t}$ and $d_{s,k}^{t}$ are polynomials of c and θ with integer coefficients, especially.

$$c_{2k,k}^{2r} = {}_rC_k\, \theta^{2r-2k},\quad c_{1,0}^{2r} = cr\theta^{2r-2},\quad c_{0,0}^{2r+1} = 2r^2\theta^{2r-1}$$

$$d_{2k,k}^{2r+1} = {}_rC_k\, \theta^{2r-2k},\quad d_{1,0}^{2r+1} = cr\theta^{2r-2},\quad d_{1,0}^{2r} = 2r(r-1)\theta^{2r-3},$$

$$\left({}_rC_k = \frac{r!}{k!(r-k)!} \right).$$

In this formula, we may replace U by V, X, or Y.

Let $K(x,\xi)$ be a homogeneous polynomial of degree t in $= (\xi_0, \xi_1, \cdots, \xi_n)$. We shall write $K^{(\gamma)}(x,\xi) = D_\xi^\gamma K(x,\xi)$.

We define $K_j(x,\xi,\eta) = \frac{1}{j!} \sum_{|\gamma|=j} \xi^\gamma K^{(\gamma)}(x,\eta)$, where $\eta = (\eta_0, \eta_1, \cdots, \eta_n)$, $\gamma = (\gamma_0, \gamma_1, \cdots, \gamma_n)$, and $|\gamma| = \gamma_0 + \gamma_1 + \cdots + \gamma_n$. So, $K(x, r\xi + s\eta) = \sum_{i=0}^t K_i(x, r\xi, s\eta) = \sum_{i=0}^t r^i s^{t-i} K_i(x,\xi,\eta)$ holds, where $r,s \in C^1$. We shall use $\partial_i = \theta_{x_i}\partial_\theta + \rho_{x_i}\partial_\rho$ $(i=0,\cdots,n)$, $\partial = (\partial_0, \partial_1, \cdots, \partial_n)$ and $D_i\partial_j = \theta_{x_i x_j}\partial_\rho + \rho_{x_i x_j}\partial_\rho$.

The relation $K(x,\partial) = K(x, \theta_x\partial_\theta + \rho_x\partial_\rho) = \sum_{i=0}^t K_i(x, \theta_x, \rho_x)\, \partial_\theta^i \partial_\rho^{t-i}$ and (F.1) lead to the following formula.

(F.2)

$$
\begin{cases}
K(x,\partial)U_\alpha^{(l)} = \sum\limits_{k=0}^{[\frac{t}{2}]} \sum\limits_{s=t-k}^{[\frac{t-1}{2}]+1} K_{k,s}^{t,1}(x,\theta,\rho)U_{\alpha-s+k}^{(l-k)} \\
\qquad + \sum\limits_{k=0}^{[\frac{t-1}{2}]} \sum\limits_{s=t-k}^{[\frac{t}{2}]-1} K_{k,s}^{t,2}(x,\theta,\rho)\partial_\theta U_{\alpha-s+k+1}^{(l-k)} \\
K(x,\partial)\partial_\theta U_\alpha^{(l)} = \sum\limits_{k=0}^{[\frac{t+1}{2}]} \sum\limits_{s=t-k}^{[\frac{t}{2}]} K_{k,s}^{t,3}(x,\theta,\rho)U_{\alpha-s+k-1}^{(l-k)}
\end{cases}
$$

$$+\sum_{k=0}^{[\frac{t}{2}]}\sum_{s=t-k}^{[\frac{t+1}{2}]} K_{k,s}^{t,4}(x,\theta,\rho)\partial_\theta U_{\alpha-s+k}^{(l-k)} \quad ,$$

([A] is a integer part of A.)

,where $K_{0,t}^{t,1}(x,\theta,\rho)=\sum_{i=0}^{[\frac{t}{2}]} K_{2i}(x,\theta_x,\rho_x)\theta^{2i}$,

$$K_{0,t-1}^{t,1}(x,\theta,\rho)=\sum_{i=1}^{[\frac{t}{2}]} ci\,\theta^{2i-2}K_{2i+1}(x,\theta_x,\rho_x)+\sum_{i=1}^{[\frac{t-1}{2}]} 2i^2\theta^{2i-1}K_{2i+1}(x,\theta_x,\rho_x),$$

$$K_{0,t}^{t,2}(x,\theta,\rho) = \sum_{i=0}^{[\frac{t-1}{2}]} K_{2i+1}(x,\theta_x,\rho_x) \quad ,$$

$$K_{0,t-1}^{t,2}(x,\theta,\rho)=\sum_{i=1}^{[\frac{t-1}{2}]} ci\,\theta^{2i-2}K_{2i+1}(x,\theta_x,\rho_x)+$$

$$+\sum_{i=2}^{[\frac{t}{2}]} 2i(i-1)\theta^{2i-3}K_{2i}(x,\theta_x,\rho_x) \quad ,$$

$$K_{k,t-k}^{t,1}(x,\theta,\rho)=\sum_{i=k}^{[\frac{t}{2}]} {}_iC_k\theta^{2i-2k}K_{2i}(x,\theta_x,\rho_x) \quad ,$$

$$K_{k,t-k}^{t,2}(x,\theta,\rho) = \sum_{i=k}^{[\frac{t-1}{2}]} {}_iC_k\theta^{2i-2k}\, K_{2i+1}(x,\theta_x,\rho_x) \quad ,$$

$$K_{0,t}^{t,3}(x,\theta,\rho)=\sum_{i=0}^{[\frac{t+1}{2}]} K_{2i+1}(x,\theta_x,\rho_x)\cdot\theta^{2i+2} \quad ,$$

$$K_{0,t-1}^{t,3}(x,\theta,\rho) = \sum_{i=1}^{[\frac{t+1}{2}]} ci\,\theta^{2i-2}K_{2i-1}(x,\theta_x,\rho_x)$$

$$+\sum_{i=1}^{[\frac{t}{2}]} 2i^2\theta^{2i-1}K_{2i}(x,\theta_x,\rho_x) \quad ,$$

$$K_{0,t}^{t,4}(x,\theta,\rho)=\sum_{i=0}^{[\frac{t}{2}]} \theta^{2i}K_{2i}(x,\theta_x,\rho_x) \quad ,$$

$$K_{0,t-1}^{t,4}(x,\theta,\rho) = \sum_{i=1}^{[\frac{t}{2}]} ci\,\theta^{2i-2}K_{2i}(x,\theta_x,\rho_x)$$

$$+\sum_{i=2}^{[\frac{t+1}{2}]} 2i(i-1)\theta^{2i-3}K_{2i-1}(x,\theta_x,\rho_x),$$

$$K_{k,t-k}^{t,3}(x,\theta,\rho)=\sum_{i=k}^{[\frac{t+1}{2}]} {}_iC_k\theta^{2i-2k}K_{2i-1}(x,\theta_x,\rho_x) \quad ,$$

$$K^{t,4}_{k,t-k}(x,\theta,\rho)=\sum_{i=k}^{[\frac{t}{2}]} {}_iC_k\theta^{2i-2k}K_{2i}(x,\theta_x,\rho_x) \quad .$$

We shall sometimes use the following formula by chain rule:

$$(F.3) \quad K(x,D)\left[w(x)W(\theta(x),\rho(x))\right] = w\cdot K(x,\partial)W$$

$$+ \sum_{i,j=0}^{n}\frac{1}{2}K^{(i,j)}(x,\partial)(D_i\partial_j)W + \sum_{i=0}^{n} D_i w\cdot K^{(i)}(x,\partial)W$$

+ (lower order term).

Now using (F.2) and (F.3), we caluculate $L(x,D)u(x)U^{(j)}_{\alpha-1}(\theta(x),\rho(x))$ and $L(x,D)g(x)\,\partial_\theta U^{(j)}_\alpha(\theta(x),\rho(x))$. We have,

$$(F.4) \quad L(x,D)uU^{(j)}_{\alpha-1}=\sum_{|\gamma|=0}^{2m}\frac{1}{\gamma!}D^\gamma u\cdot L^{(\gamma)}(x,D)U^{(j)}_{\alpha-1}$$

$$=\sum_{|\gamma|=0}^{2m}\frac{1}{\gamma!}D^\gamma u\cdot\Big[\overset{\circ}{L}{}^{(\gamma)}(x,\partial)+\sum_{i,j=0}^{n}\frac{1}{2}\overset{\circ}{L}{}^{(\gamma)(i,j)}(x,\partial)(D_i\partial_j)$$

$$+ L_s(x,\partial) + \cdots\Big]\, U^{(j)}_{\alpha-1}$$

$$=\sum_{k=0}^{m}\sum_{\nu=0}^{2m-2k} {}^1L_{\nu,k}(u)U^{(j-k)}_{\alpha-\nu-1} + \sum_{k=0}^{m-1}\sum_{\nu=1}^{2m-2k} {}^2L_{\nu,k}(u)\,\partial_\theta U^{(j-k)}_{\alpha-\nu} \quad .$$

$$L(x,D)g\,\partial_\theta U^{(j)}_\alpha=\sum_{k=0}^{m}\sum_{\nu=-1}^{2m-2k} {}^3L_{\nu,k}(g)U^{(j-k)}_{\alpha-\nu-1} +$$

$$\sum_{k=0}^{m}\sum_{\nu=0}^{2m-2k} {}^4L_{\nu,k}(g)\partial_\theta U^{(j-k)}_{\alpha-\nu} \quad ,$$

where ${}^hL_{\nu,k}={}^hL_{\nu,k}(x,\theta,\rho,D)\in L^{2m-2k-\nu}(\Omega)$ $(h=1,2,3,4)$ and especially ${}^3L_{-1,0}=0$.

We see also the following relations from the above.

$$(F.5)\begin{cases} {}^{1}\overset{\circ}{L}_{\nu,k}= \sum_{|\gamma|=2m-2k-\nu} \overset{\circ}{L}{}^{(\gamma)}_{2k}(x,\theta_x,\rho_x)\frac{D^\gamma}{\gamma!} & \mathrm{mod.}\ \theta^2 , \\ {}^{2}\overset{\circ}{L}_{\nu,k}= \sum_{|\gamma|=2m-2k-\nu} \overset{\circ}{L}{}^{(\gamma)}_{2k+1}(x,\theta_x,\rho_x)\frac{D^\gamma}{\gamma!} & \mathrm{mod.}\ \theta^2 , \\ {}^{3}\overset{\circ}{L}_{\nu,k}= \sum_{|\gamma|=2m-2k-\nu} \overset{\circ}{L}{}^{(\gamma)}_{2k-1}(x,\theta_x,\rho_x)\frac{D^\gamma}{\gamma!} & \mathrm{mod.}\ \theta^2\ (k\geq 1) , \\ {}^{3}\overset{\circ}{L}_{\nu,0}= \theta^2\left\{ \sum_{|\gamma|=2m-\nu} \overset{\circ}{L}{}^{(\gamma)}_{1}(x,\theta_x,\rho_x)\frac{D^\gamma}{\gamma!}\right\} & \mathrm{mod.}\ \theta^4 , \\ {}^{4}\overset{\circ}{L}_{\nu,k}= \sum_{|\gamma|=2m-2k-\nu} \overset{\circ}{L}{}^{(\gamma)}_{2k}(x,\theta_x,\rho_x)\frac{D^\gamma}{\gamma!} & \mathrm{mod.}\ \theta^2 , \end{cases}$$

where ${}^{h}\overset{\circ}{L}_{\nu,k}$ are (is) the principal part of ${}^{h}L_{\nu,k}$.

More precisely for $k=0$ and $\nu=2m$ or $2m-1$, we have:

$$(F.6)\begin{cases} {}^{1}L_{2m,0}= \sum_{i=0}^{m}\overset{\circ}{L}_{2i}(x,\theta_x,\rho_x)\theta^{2i} , \\ {}^{2}L_{2m,0}= \sum_{i=0}^{m-1}\overset{\circ}{L}_{2i+1}(x,\theta_x,\rho_x)\theta^{2i} , \\ {}^{3}L_{2m,0}= \sum_{i=0}^{m-1}\overset{\circ}{L}_{2i+1}(x,\theta_x,\rho_x)\theta^{2i+2} , \\ {}^{4}L_{2m,0}= \sum_{i=0}^{m}\overset{\circ}{L}_{2i}(x,\theta_x,\rho_x)\theta^{2i} , \end{cases}$$

and

$$(F.7)\begin{cases} {}^{1}L_{2m-1,0}(x,\theta,\rho,D)= M+R_0+N_c+R_2+\frac{1}{2}\left\{\sum_{\mu,\nu=0}^{n}\rho_{x_\mu x_\nu}S_0^{(\mu,\nu)} + \theta_{x_\mu x_\nu}\theta^2 S_1^{(\mu,\nu)}\right\} , \\ {}^{2}L_{2m-1,0}(x,\theta,\rho,D)= L+R_1+N_c''+R_2'+\frac{1}{2}\left\{\sum_{\mu,\nu=0}^{n}\theta_{x_\mu x_\nu}S_0^{(\mu,\nu)} + \rho_{x_\mu x_\nu}S_1^{(\mu,\nu)}\right\} , \\ {}^{3}L_{2m-1,0}(x,\theta,\rho,D)= \theta^2 L+\theta^2\left\{R_1+\frac{1}{2}\sum_{\mu,\nu=0}^{n}\theta_{x_\mu x_\nu}S_0^{(\mu,\nu)}\right. \end{cases}$$

$$+ \rho_{x_\mu x_\nu} S_1^{(\mu,\nu)}\Big\} + N_c' + R_3' \quad ,$$

$$^4L_{2m-1,0}(x,\theta,\rho,D) = M + R_0 + N_c + R_3 + \frac{1}{2}\Big\{ \sum_{\mu,\nu=0}^{n} \rho_{x_\mu x_\nu} S_0^{(\mu,\nu)}$$

$$+ \theta_{x_\mu x_\nu} S_1^{(\mu,\nu)} \Big\} \quad ,$$

where $L = \sum_{|\gamma|=1} \Big\{ \sum_{i=1}^{m-1} \overset{\circ}{L}{}^{(\gamma)}_{2i+1}(x,\theta_x,\rho_x)\theta^{2i} \Big\} D^\gamma \quad ,$

$$M = \sum_{|\gamma|=1} \Big\{ \sum_{i=0}^{m-1} \overset{\circ}{L}{}^{(\gamma)}_{2i}(x,\theta_x,\rho_x)\theta^{2i} \Big\} D^\gamma \quad ,$$

$$R_0 = \sum_{i=0}^{m-1} \overset{\circ}{L}_{s,2i}(x,\theta_x,\rho_x)\theta^{2i} \quad ,$$

$$R_1 = \sum_{i=0}^{m-1} \overset{\circ}{L}_{s,2i+1}(x,\theta_x,\rho_x)\theta^{2i} \quad ,$$

$$R_2 = \sum_{i=2}^{m} \overset{\circ}{L}_{2i-1}(x,\theta_x,\rho_x) \cdot 2(i-1)^2 \cdot \theta^{2i-3} \quad ,$$

$$R_2' = \sum_{i=2}^{m} \overset{\circ}{L}_{2i}(x,\theta_x,\rho_x) \cdot 2i(i-1) \cdot \theta^{2i-3} \quad ,$$

$$R_3 = \sum_{i=2}^{m} \overset{\circ}{L}_{2i-1}(x,\theta_x,\rho_x) \cdot 2i(i-1) \cdot \theta^{2i-3} \quad ,$$

$$R_3' = \sum_{i=1}^{m} \overset{\circ}{L}_{2i}(x,\theta_x,\rho_x) \cdot 2i^2 \, \theta^{2i-1} \quad ,$$

$$N_c = \sum_{i=1}^{m} \overset{\circ}{L}_{2i}(x,\theta_x,\rho_x) \cdot ci \cdot \theta^{2i-2} \quad ,$$

$$N_c' = \sum_{i=1}^{m} \overset{\circ}{L}_{2i-1}(x,\theta_x,\rho_x) \, ci \cdot \theta^{2i-2} \quad ,$$

$$N_c'' = \sum_{i=1}^{m} \overset{\circ}{L}_{2i-1}(x,\theta_x,\rho_x) \cdot c(i-1) \cdot \theta^{2i-4} \quad ,$$

$$S_0^{(\mu,\nu)} = \sum_{i=0}^{m-1} L_{2i}^{(\mu,\nu)}(x,\theta_x,\rho_x)\theta^{2i} \quad ,$$

$$S_1^{(\mu,\nu)} = \sum_{i=1}^{m-1} L_{2i-1}^{(\mu,\nu)}(x,\theta_x,\rho_x)\theta^{2i-2} \quad .$$

We remark, in the above, next relations:

$$L_0(x,\theta_x,\rho_x)=L(x,\rho_x) \;,\quad L_1(x,\theta_x,\rho_x)=\sum_{\mu=0}^{n}\theta_{x_\mu}L^{(\mu)}(x,\rho_x) \;, \text{ and}$$

$$L_2(x,\theta_x,\rho_x)=\frac{1}{2}\sum_{\mu,\nu=0}^{n}\theta_{x_\mu}\cdot\theta_{x_\nu}L^{(\mu,\nu)}(x,\rho_x) \;.$$

We remark also that in these formulae (F.1), (F.2), (F.3) and (F.4) obtained above , we may replace U by V, X and Y .

§. 3 Construction of the formal solution.

Taking account of the principle of the superposition, we have only to solve the following Cauchy problem with the special initial data:

$$\begin{cases} L(x,D)u(x)=0 \\ D_0^h u(0,x')=w_h(x'')k_{-1}(x_1) \quad (h=0,\cdots,2m-1) \end{cases}$$

where $x''=(x_2,\cdots,x_n)$ and $w_h(x'')$ are holomorphic functions of x'' in the neighbourhood of $0\in C^{n-1}$.

We seek the formal solution of the form in Theorem 1.1 . Namely we determine the coefficients $u_{\alpha,\beta}^{(q)}$, $g_{\alpha,\beta}^{(q)}$, $v_{\alpha,\beta}^{(q)}$, $h_{\alpha,\beta}^{(q)}$ and the auxiliary phase functions θ_β, ρ_β ,and show the convergence of this formal solution . First to determine these coefficients and auxiliary phase functions , we substitute this formal solution in $L(x,D)u(x)=0$, and using the formulae obtained in the preceeding section and the relations $\partial_\rho X_{\alpha,\beta}^{(q)} = X_{\alpha-1,\beta}^{(q)}$ and $\partial_\rho Y_{\alpha,\beta}^{(q)}=Y_{\alpha-1,\beta}^{(q)}$, we have,

$$
\begin{aligned}
L(x,D)u(x)= \sum_{\beta=1}^{m} \sum_{\alpha} \sum_{j} \Bigg[\Bigg\{ \sum_{k=0}^{m} \sum_{\nu=-1}^{2m-2k} \Big(& {}^{1}L_{\nu,k,\beta}(u^{(j+k)}_{\alpha+\nu,\beta}) \\
+ {}^{3}L_{\nu,k,\beta}(g^{(j+k)}_{\alpha+\nu,\beta}) \Big) \Bigg\} X^{(j)}_{\alpha-1,\beta} & + \Bigg\{ \sum_{k=0}^{m} \sum_{\nu=0}^{2m-2k} \Big({}^{2}L_{\nu,k,\beta}(u^{(j+k)}_{\alpha+\nu,\beta}) \\
+ {}^{4}L_{\nu,k,\beta}(g^{(j+k)}_{\alpha+\nu,\beta}) \Big) \Bigg\} \partial_{\theta} X^{(j)}_{\alpha,\beta} & + \Bigg\{ \sum_{k=0}^{m} \sum_{\nu=-1}^{2m-2k} \Big({}^{1}L_{\nu,k,\beta}(v^{(j+k)}_{\alpha+\nu,\beta}) \\
+ {}^{3}L_{\nu,k,\beta}(h^{(j+k)}_{\alpha+\nu,\beta}) \Big) \Bigg\} Y^{(j)}_{\alpha-1,\beta} & + \Bigg\{ \sum_{k=0}^{m} \sum_{\nu=0}^{2m-2k} \Big({}^{2}L_{\nu,k,\beta}(v^{(j+k)}_{\alpha+\nu,\beta}) \\
+ {}^{4}L_{\nu,k,\beta}(h^{(j+k)}_{\alpha+\nu,\beta}) \Big) \Bigg\} \partial_{\theta} Y^{(j)}_{\alpha,\beta} \Bigg] & =0 \quad ,
\end{aligned}
$$

where ${}^{h}L_{\nu,k,\beta} = {}^{h}L_{\nu,k}\ (x, \theta_\beta, \rho_\beta, D)$ $(h=1,2,3,4)$ and

${}^{1}L_{-1,k,\beta} = {}^{2}L_{\nu,m,\beta} = {}^{2}L_{0,k,\beta} = 0$.

We set the coefficients of $X^{(j)}_{\alpha-1,\beta}$, $\partial_\theta X^{(j)}_{\alpha,\beta}$, $Y^{(j)}_{\alpha-1,\beta}$ and $\partial_\theta Y^{(j)}_{\alpha,\beta}$ equal to zero . And especially we set ${}^{1}L_{2m,0,\beta} = {}^{2}L_{2m,0,\beta} = {}^{3}L_{2m,0,\beta} = {}^{4}L_{2m,0,\beta} = 0$ which are non-linear partial differential equations of first order in θ_β and ρ_β , namely so-called eikonal equations , so that we can determine these auxiliary phase functions θ_β and ρ_β ,(we shall study these non-linear partial differential equations in the next section). Thus we have reached the systems of the transport equations which determine the coefficients $u^{(q)}_{\alpha,\beta}$, $g^{(q)}_{\alpha,\beta}$, $v^{(q)}_{\alpha,\beta}$ and $h^{(q)}_{\alpha,\beta}$.

$$
(\mathrm{T.E.})\begin{cases}
(\mathrm{i})\begin{cases}
{}^{3}L_{2m-1,0,\beta}(g^{(j)}_{\alpha+2m-1,\beta}) + {}^{1}L_{2m-1,0,\beta}(u^{(j)}_{\alpha+2m-1,\beta}) \\
\quad = -\sum_{k=1}^{m}\sum_{\nu=-1}^{2m-2k}\left\{{}^{1}L_{\nu,k,\beta}(u^{(j+k)}_{\alpha+\nu,\beta}) + {}^{3}L_{\nu,k,\beta}(g^{(j+k)}_{\alpha+\nu,\beta})\right\} \\
\qquad - \sum_{\nu=-1}^{2m-2}{}'\left\{{}^{1}L_{\nu,0,\beta}(u^{(j)}_{\alpha+\nu,\beta}) + {}^{3}L_{\nu,0,\beta}(g^{(j)}_{\alpha+\nu,\beta})\right\}, \\
{}^{2}L_{2m-1,0,\beta}(u^{(j)}_{\alpha+2m-1,\beta}) + {}^{4}L_{2m-1,0,\beta}(g^{(j)}_{\alpha+2m-1,\beta}) \\
\quad = -\sum_{k=1}^{m}\sum_{\nu=0}^{2m-2k}\left\{{}^{2}L_{\nu,k,\beta}(u^{(j+k)}_{\alpha+\nu,\beta}) + {}^{4}L_{\nu,k,\beta}(g^{(j+k)}_{\alpha+\nu,\beta})\right\} \\
\qquad - \sum_{\nu=-1}^{2m-2}\left\{{}^{2}L_{\nu,0,\beta}(u^{(j)}_{\alpha+\nu,\beta}) + {}^{4}L_{\nu,0,\beta}(g^{(j)}_{\alpha+\nu,\beta})\right\},
\end{cases} \\
(\mathrm{ii})\begin{cases}
{}^{3}L_{2m-1,0,\beta}(h^{(j)}_{\alpha+2m-1,\beta}) + {}^{1}L_{2m-1,0,\beta}(v^{(j)}_{\alpha+2m-1,\beta}) \\
\quad = -\sum_{k=1}^{m}{}'\sum_{\nu=-1}^{2m-2k}\left\{{}^{1}L_{\nu,k,\beta}(v^{(j+k)}_{\alpha+\nu,\beta}) + {}^{3}L_{\nu,k,\beta}(h^{(j+k)}_{\alpha+\nu,\beta})\right\} \\
\qquad - \sum_{\nu=-1}^{2m-2}\left\{{}^{1}L_{\nu,0,\beta}(v^{(j)}_{\alpha+\nu,\beta}) + {}^{3}L_{\nu,0,\beta}(h^{(j)}_{\alpha+\nu,\beta})\right\}, \\
{}^{2}L_{2m-1,0,\beta}(v^{(j)}_{\alpha+2m-1,\beta}) + {}^{4}L_{2m-1,0,\beta}(h^{(j)}_{\alpha+2m-1,\beta}) \\
\quad = -\sum_{k=1}^{m}{}'\sum_{\nu=0}^{2m-2k}\left\{{}^{2}L_{\nu,k,\beta}(v^{(j+k)}_{\alpha+\nu,\beta}) + {}^{4}L_{\nu,k,\beta}(h^{(j+k)}_{\alpha+\nu,\beta})\right\} \\
\qquad - \sum_{\nu=-1}^{2m-2}\left\{{}^{2}L_{\nu,0,\beta}(v^{(j)}_{\alpha+\nu,\beta}) + {}^{4}L_{\nu,0,\beta}(h^{(j)}_{\alpha+\nu,\beta})\right\}.
\end{cases}
\end{cases}
$$

On the other hand, from the initial data we substitute the formal solution $u(x)$ in $D_0^h u(x)\big|_{x_0=0}$ and caluculate this ,using the formulae (F.4), (F.5) and (F.6) which are valid under the replace -ment of $L(x,D)$ by D_0^h , we have,

$$D_0^h u(x)\Big|_{x_0=0} = \sum_{\beta=1}^{m} \sum_{\alpha} \sum_{j=0}^{m} \Bigg[\Big\{\sum_{k=0}^{[\frac{h}{2}]} \sum_{\nu=0}^{h-2k} {}^1M^{\ h}_{\nu,k,\beta}(u^{(j)}_{\alpha,\beta}) X^{(j-k)}_{\alpha-\nu-1,\beta}\Big\} +$$
$$\Big\{\sum_{k=0}^{[\frac{h-1}{2}]} \sum_{\nu=0}^{h-2k} {}^2M^{\ h}_{\nu,k,\beta}(u^{(j)}_{\alpha,\beta})\, \partial_\theta X^{(j-k)}_{\alpha-\nu,\beta}\Big\} + \Big\{\sum_{k=0}^{[\frac{h+1}{2}]} \sum_{\nu=-1}^{h-2k} {}^3M^{\ h}_{\nu,k,\beta}(g^{(j)}_{\alpha,\beta}) X^{(j-k)}_{\alpha-\nu-1,\beta}\Big\}$$
$$+\Big\{\sum_{k=0}^{[\frac{h}{2}]} \sum_{\nu=0}^{h-2k} {}^4M^{\ h}_{\nu,k,\beta}(g^{(j)}_{\alpha,\beta})\partial_\theta X^{(j-k)}_{\alpha-\nu,\beta}\Big\} + \Big\{\sum_{k=0}^{[\frac{h}{2}]} \sum_{\nu=0}^{h-2k} {}^1M^{\ h}_{\nu,k,\beta}(v^{(j)}_{\alpha,\beta}) Y^{(j-k)}_{\alpha-\nu-1,\beta}\Big\}$$
$$+\Big\{\sum_{k=0}^{[\frac{h-1}{2}]} \sum_{\nu=0}^{h-2k} {}^2M^{\ h}_{\nu,k,\beta}(v^{(j)}_{\alpha,\beta})\, \partial_\theta Y^{(j-k)}_{\alpha-\nu,\beta}\Big\} + \Big\{\sum_{k=0}^{[\frac{h+1}{2}]} \sum_{\nu=-1}^{h-2k} {}^3M^{\ h}_{\nu,k,\beta}(h^{(j)}_{\alpha,\beta}) Y^{(j-k)}_{\alpha-\nu-1,\beta}\Big\}$$
$$+\Big\{\sum_{k=0}^{[\frac{h}{2}]} \sum_{\nu=0}^{h-2k} {}^4M^{\ h}_{\nu,k,\beta}(h^{(j)}_{\alpha,\beta})\partial_\theta Y^{(j-k)}_{\alpha-\nu,\beta}\Big\}\Bigg]\Bigg|_{x_0=0} ,$$

where ${}^pM^{\ h}_{\nu,k,\beta}(p=1,2,3,4)$ is a linear ordinary differential operators in D_0 of order $h-\nu-2k$ and especially ${}^1M^{\ h}_{h,0,\beta}={}^4M^{\ h}_{h,0,\beta}$ $= (\rho_{\beta x_0}(0,x'))^h$, ${}^1\mathring{M}^{\ h}_{\nu,k,\beta}={}^4\mathring{M}^{\ h}_{\nu,k,\beta}$, ${}^2M^{\ h}_{h,0,\beta}=h(\rho_{\beta_{x_0}}(0,x'))^{h-1}\sigma_\beta(0,x')$ and ${}^3M^{\ h}_{-1,0,\beta}={}^4M^{\ h}_{h,0,\beta}=0$ (as for $\sigma_\beta(0,x')$, see §4) .

These ordinary differential operators are determined only by h and θ_β , ρ_β , and have holomorphic coefficients in x' .

We have :
$$D_0^h u(x)\Big|_{x_0=0} = \sum_{\beta=1}^{m} \sum_{\alpha} \sum_{j} \Bigg(\Big[\sum_{k=0}^{[\frac{h+1}{2}]} \sum_{\nu=-1}^{h-2k} \Big\{{}^1M^{\ h}_{\nu,k,\beta}(u^{(j)}_{\alpha,\beta}) + {}^3M^{\ h}_{\nu,k,\beta}(g^{(j)}_{\alpha,\beta})\Big\}\Big] X^{(j-k)}_{\alpha-\nu-1,\beta} + \Big[\sum_{k=0}^{[\frac{h}{2}]} \sum_{\nu=0}^{h-2k} \Big\{{}^2M^{\ h}_{\nu,k,\beta}(v^{(j)}_{\alpha,\beta}) +$$
$${}^4M^{\ h}_{\nu,k,\beta}(h^{(j)}_{\alpha,\beta})\Big\}\Big]\, \partial_\theta Y^{(j-k)}_{\alpha-\nu,\beta}\Bigg)\Bigg|_{x_0=0} .$$

Setting the coefficients of $X^{(0)}_{\alpha,\beta}$ and $\partial_\theta Y^{(0)}_{\alpha,\beta}$ equal to zero, we obtain the following systems of linear equations:

$$(\mathrm{I.E})\quad \sum_{\beta=1}^{m} \sum_{k=0}^{\left[\frac{h+1}{2}\right]} \sum_{\nu=-1}^{h-2k} \left\{ {}^{1}M^{\;h}_{\nu,k,\beta} (u^{(k)}_{\alpha+\nu+1,\beta}) + {}^{3}M^{\;h}_{\nu,k,\beta} (g^{(k)}_{\alpha+\nu+1,\beta}) \right\} + \sum_{k=0}^{\left[\frac{h}{2}\right]} \sum_{\nu=0}^{h-2k} \left\{ {}^{2}M^{\;h}_{\nu,k,\beta} (v^{(k)}_{\alpha+\nu,\beta}) + {}^{4}M^{\;h}_{\nu,k,\beta} (h^{(k)}_{\alpha+\nu,\beta}) \right\} \Bigg|_{x_0=0}$$

$$= \begin{cases} w_h(x'') & \text{for } \alpha = -1 + h+1 \quad , \\ 0 & \text{otherwise} \quad . \end{cases}$$

Remark that the determinant of the following $2m \times 2m$ matrix

$$\begin{vmatrix} 1 & , 0 & ,\cdots, & 1 & , 0 \\ \gamma_1 & , 1 & ,\cdots, & \gamma_m & , 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \gamma_1^{2m-1} & ,(2m-1)\gamma_1^{2m-2} & ,\cdots, & \gamma_m^{2m-1} & ,(2m-1)\gamma_m^{2m-2} \end{vmatrix}$$

$$(\gamma_\beta = f_{\beta x_0}(0,x') \quad)$$

does not vanish if γ_β ($\beta = 1, \cdots, m$) are mutually distinct,

(in § 4 we shall see γ_β are mutually distinct) .

From the remark described above, we can solve the system of

2m linear equations with respect to $(u^{(0)}_{\alpha+2m-1,\beta} + h^{(0)}_{\alpha+2m-2,\beta})(0,x')$

and $v^{(0)}_{\alpha+2m-2,\beta}(0,x')$ ($\beta = 1,\cdots,m$) and then we obtain

Lemma 3.1 $(u^{(0)}_{\alpha+2m-1,\beta}+h^{(0)}_{\alpha+2m-2,\beta})(0,x')$ and $v^{(0)}_{\alpha+2m-2,\beta}(0,x')$ are represented in the following form :

$$\sum_{\beta=1}^{m}\Bigg[\Bigg\{\sum_{k=1}^{m}\sum_{\mu=2k}^{2m-2} {}^{1}\tilde{N}^{(k)}_{\mu-2k,\beta}(u^{(k)}_{\alpha+2m-1-\mu,\beta})+{}^{4}\tilde{N}^{(k)}_{\mu-2k,\beta}(h^{(k)}_{\alpha+2m-2-\mu,\beta})+$$
$${}^{3}N^{(k)}_{\mu-2k,\beta}(g^{(k)}_{\alpha+2m-2-\mu,\beta})+{}^{2}N^{(k)}_{\mu-2k,\beta}(v^{(k)}_{\alpha+2m-2-\mu,\beta})\Bigg\}+$$
$$\sum_{\mu=1}^{2m-2}\Bigg\{{}^{1}N^{(0)}_{\mu,\beta}(u^{(0)}_{\alpha+2m-1-\mu,\beta}+h^{(0)}_{\alpha+2m-2-\mu,\beta})+{}^{1}\tilde{N}^{(0)}_{\mu,\beta}u^{(0)}_{\alpha+2m-1-\mu,\beta}+$$
$${}^{4}\tilde{N}^{(0)}_{\mu,\beta}h^{(0)}_{\alpha+2m-2-\mu,\beta}+{}^{3}N^{(0)}_{\mu,\beta}g^{(0)}_{\alpha+2m-2-\mu,\beta}+{}^{2}N^{(0)}_{\mu,\beta}v^{(0)}_{\alpha+2m-2-\mu,\beta}\Bigg\}\Bigg]\Bigg|_{x_0=0}$$

where ${}^{h}N^{(k)}_{\mu,\beta}$ are linear ordinary differential operators in D_0 of order μ, and ${}^{3}N^{(0)}_{\mu,\beta}$, ${}^{1}\tilde{N}^{(k)}_{\mu,\beta}$ and ${}^{4}\tilde{N}^{(k)}_{\mu,\beta}$ are linear ordinary differential operators in D_0 of order $\mu-1$. These linear ordinary differential operators are determined only by $\theta_\beta(x)$ and $f_\beta(x)$, and have holomorphic coefficients in x'.

We are to determine θ_β and f_β from the so-called eikonal equations that are to be studied in the next section, and these coefficients from the transport equations that are to be studied in §5 precisely.

§4. Phase functions and auxiliary phase functions.

In this section, we study eikonal equations

$$(4.1) \qquad {}^{h}L_{2m,0,\beta}(x, \theta_{\beta x}, \rho_{\beta x})=0 \qquad (h=1,2,3,4)$$

and some properties of $\theta_\beta(x)$ and $\rho_\beta(x)$.

From the definition of ${}^{h}L_{2m,0,\beta}$, (F,6), we write again:

$$\begin{cases} {}^{1}L_{2m,0,\beta} = {}^{4}L_{2m,0,\beta} = \sum_{i=0}^{m} \overset{o}{L}_{2i}\,(x, \theta_{\beta x}, \rho_{\beta x})\, \theta_\beta^{2i}=0 \\ {}^{2}L_{2m,0,\beta} = \sum_{i=0}^{m-1} \overset{o}{L}_{2i+1}\,(x, \theta_{\beta x}, \rho_{\beta x})\, \theta_\beta^{2i}=0 \\ {}^{3}L_{2m,0,\beta} = \theta_\beta^{2} \cdot {}^{2}L_{2m,0,\beta} =0 . \end{cases}$$

By the homogeneity of $\overset{o}{L}_j\,(x, r\xi, \eta) = r^j \overset{o}{L}_j(x, \xi, \eta)$,

the above equations lead to

$$ {}^{1}L_{2m,0,\beta} = \sum_{i=0}^{m} \overset{o}{L}_{2i}\,(x,\ \pm\theta_\beta \cdot \theta_{\beta x}\ ,\ \rho_{\beta x})=0$$

$$\theta_\beta \cdot {}^{2}L_{2m,0,\beta} = \sum_{i=0}^{m-1} \overset{o}{L}_{2i+1}\,(x,\ \pm\theta_\beta \cdot \theta_{\beta x},\ \rho_{\beta x})=0 .$$

Adding these two equations and taking account of the relations

$$\overset{o}{L}(x, \xi+\eta) = \sum_{j=0}^{2m} \overset{o}{L}_j\,(x, \xi, \eta) \quad \text{and} \quad \theta_\beta \cdot \theta_{\beta x} = \tfrac{1}{2}(\theta_\beta^{2})_x ,$$

we obtain the ordinary eikonal equations.

$$ {}^1L_{2m,0,\beta} \pm \theta_\beta \cdot {}^2L_{2m,0,\beta} = \overset{\circ}{L}\left(x, \left(\rho_\beta \pm \frac{\theta_\beta^2}{2}\right)_x\right)=0 $$

Setting $\varphi_\beta^{\pm}(x)= \rho_\beta(x) \pm \frac{1}{2}(\theta_\beta(x))^2$, we rewrite this equation in the familiar form: $\overset{\circ}{L}(x, \varphi^{\pm}_{\beta\,x})=0$.

First we solve this Cauchy problem $\overset{\circ}{L}(x, \varphi^{\pm}_{\beta\,x})=0$ with the initial data $\varphi_\beta^{\pm}(0,x')=x_1$, and then we get $\theta_\beta(x)$ and $\rho_\beta(x)$ by $\varphi_\beta^{\pm}(x)= \rho_\beta(x)\pm \frac{1}{2}(\theta_\beta(x))^2$ and some properties of these functions.

Proposition 4.1

In a certain neighbourhood of $0\in C^{n+1}$, there exist the holomorphic solutions $\varphi_\beta^{\pm}(x)$ ($\beta=1,\cdots,m$) of the Cauchy problem:

$$ \begin{cases} \overset{\circ}{L}(x, \varphi^{\pm}_{\beta\,x})=0 \\ \varphi_\beta^{\pm}(0,x')=x_1 \quad \text{and} \quad \varphi^{\pm}_{\beta\, x_0}(0)= \lambda_\beta \end{cases} $$

Precisely speaking, $\varphi_\beta^{\pm}(x)$ are represented in the form

$$ \varphi_\beta^{\pm}(x)= \rho_\beta(x) \pm \frac{1}{2}(\theta_\beta(x))^2 . $$

$\theta_\beta(x)$ and $\rho_\beta(x)$ are holomorphic functions in a neighbourhood of $0 \in C^{n+1}$, and satisfy (4.1) respectively.

Moreover $\theta_\beta(x)$ are expressed as follows,

$$\theta_\beta(x) = x_0\, \sigma_\beta(x) \qquad \left(\sigma_\beta(0,x') = \left(\frac{\sqrt{Q(e_\beta)}}{P^{(0)}(e_\beta)} \right)^{\frac{1}{2}} \neq 0 \right)$$

And $\rho_\beta(x)$ are expressed as follows,

$$\rho_\beta(x) = x_1 + x_0\, \lambda_\beta(x') + x_0^2\, \tau_\beta(x)$$

$$\left(\tau_\beta(0,x') = \frac{-1}{P^{(0)}(e_\beta)} \left\{ P_{x_i}(e_\beta) + \sum_{i=1}^{n} \lambda_{\beta x_i}(x') P^{(i)}(e_\beta) \right\} \right).$$

We call $\varphi_\beta^{\pm}(x)$ phase functions and $\theta_\beta(x)$, $\rho_\beta(x)$ auxiliary phase functions, respectively.

§ 5 Some properties of ${}^{h}L_{\nu,k,\beta}$.

For the construction of the formal solution, we have to determine the coefficients $u^{(q)}_{\alpha,\beta}$, $g^{(q)}_{\alpha,\beta}$, $v^{(q)}_{\alpha,\beta}$ and $h^{(q)}_{\alpha,\beta}$ by solving systems of the transport equations (T.E.). In this section we study transport operators ${}^{h}L_{2m-1,0,\beta}$ (h=1,2,3,4) which govern (T.E.), and other important operators, ${}^{3}L_{\nu,0,\beta}$ and ${}^{1}L_{2m-2,1,\beta}$. First we recall the definition of the transport operators ${}^{h}L_{2m-1,0,\beta}$ (h=1,2,3,4). We rewrite (F.7) again in the following form :

$$
\begin{aligned}
{}^{1}L_{2m-1,0,\beta} &= M_\beta + \theta_\beta \cdot m_{1\beta}(x) \qquad &,\\
{}^{2}L_{2m-1.0,\beta} &= L_\beta + m_{2\beta}(x) \qquad &,\\
{}^{3}L_{2m-1,0,\beta} &= \theta_\beta^{2} \cdot L_\beta + \theta_\beta \cdot m_{0\beta}(x') + \theta_\beta^{2} \cdot m_{3\beta}(x) \qquad &,\\
{}^{4}L_{2m-1.0,} &= M_\beta + \theta_\beta \, m_{4\beta}(x) \qquad &,
\end{aligned}
$$

where $L = \sum_{|\gamma|=1} \left\{ \sum_{i=0}^{m} \overset{o}{L}{}^{(\gamma)}_{2i+1}(x,\theta_{\beta x},\rho_{\beta x})\theta_\beta^{2i} \right\} D^\gamma$ and

$$M = \sum_{|\gamma|=1} \left\{ \sum_{i=0}^{m-1} \overset{o}{L}{}^{(\gamma)}_{2i}(x,\theta_{\beta x},\rho_{\beta x})\theta_\beta^{2i} \right\} D^\gamma .$$

These transport operators have next properties which play an important role in the determinations of the coefficients.

Noting $e_\beta = (0,x',\rho_{\beta x}(0,x'))$, we have :

Lemma 5.1 (i) $L_\beta \Big|_{x_0=0} = 2\sigma_\beta(0,x')P^{(0)}(e_\beta)\left\{\sum_{j=0}^{n} P^{(j)}(e_\beta)D_j\right\}$,

(ii) $M_\beta = x_0 P_\beta(x')\left\{\sum_{j=0}^{n} P^{(j)}(e_\beta)D_j\right\} + x_0^2 \cdot M'_\beta$,

(where $P(x,\rho_{\beta x}(x)) = x_0 P'_\beta(x)$ and $P'_\beta(0,x') = P_\beta(x')$)

(iii) $m_{0\beta}(x') = 2\left\{\sigma_\beta(0,x')P^{(0)}(e_\beta)\right\}^2 \neq 0$,

(iv) ${}^1L_{2m-1,0,\beta}\Big|_{x_0=0} = {}^4L_{2m-1,0,\beta}\Big|_{x_0=0} = 0$.

In this lemma , Assumptions (P) and (Q) guarantee that the initial surface $x_0=0$ is non-characteristic for L_β ,and Assumption (L_s) is (iv) itself. By this lemma we can express transport operators in the following form:

$$
(\mathrm{T.O.})\begin{cases}
{}^1L_{2m-1,0,\beta} = \theta_\beta \cdot \widetilde{M}_\beta & , \\
{}^2L_{2m-1,0,\beta} = L_\beta + m_{2\beta}(x) & , \\
{}^3L_{2m-1.0,\beta} = \theta_\beta \cdot \widetilde{L}_\beta & , \\
\quad (\text{where } \widetilde{L}_\beta = \theta_\beta \cdot L_\beta + m_{0\beta}(x') + \theta_\beta \cdot m_{3\beta}(x) \quad) & \\
{}^4L_{2m-1,0,\beta} = \theta_\beta \cdot \widetilde{M}'_\beta & .
\end{cases}
$$

On the other hand, ${}^{1}L_{2m-2,1,\beta}$ are functions which play an important role in the determination of the initial data of the transport equations (T.E.). We know the next property of ${}^{1}L_{2m-2,1,\beta}$.

Lemma 5.2 $\quad {}^{1}L_{2m-2,1,\beta}\Big|_{x_0=0} = m_{0\beta}(x') = \left[\sigma_\beta(0,x')P^{(0)}(e_\beta)\right]^2 \neq 0.$

In the estimation of the right hand sides of the transport equations (T.E.) for the proof of the convergence of the formal solution, ${}^{3}L_{\nu,0,\beta}$ plays an important role. We see the following expression of ${}^{3}L_{\nu,0,\beta}$.

Lemma 5.3 $\quad {}^{3}L_{\nu,0,\beta} \equiv \sum_{|\gamma|=2m-\nu-1} c_\beta \overset{\circ}{L}{}^{(\gamma)}_{1}(x,\theta_{\beta x},\rho_{\beta x})\frac{D^\gamma}{\gamma!}$

$+$(operators of order $2m-\nu-2$) $\quad$ mod. x_0

§.6 Determination of $u^{(q)}_{\alpha,\beta}$, $g^{(q)}_{\alpha,\beta}$, $v^{(q)}_{\alpha,\beta}$ and $h^{(q)}_{\alpha,\beta}$.

In §.3, we obtained the transport equations (T.E.), that is ,the first order system by which $u^{(q)}_{\alpha,\beta}$, $g^{(q)}_{\alpha,\beta}$, $v^{(q)}_{\alpha,\beta}$ and $h^{(q)}_{\alpha,\beta}$ are determined. First we remark these first order system are composed of the first order systems of the same form:

$$
\text{(F.S.)} \quad \begin{cases} \theta_\beta \cdot \tilde{L}_\beta g + \theta_\beta \cdot \tilde{M}_\beta u = D(x) \\ (L_\beta + m_{2\beta})u + \theta_\beta \cdot \tilde{M}'_\beta g = E(x) \end{cases}
$$

We note that for the solvability of this first order system of u and g , the following condition is necessary:

$$
D(x)\big|_{x_0=0} = 0
$$

We shall use next notations.

$$
R\left[f(x)\right] = \frac{1}{x_0}\left(f(x) - f(0,x') \right) ,
$$

$$
R_\beta\left[f(x)\right] = \frac{1}{\theta_\beta(x)}\left(f(x) - f(0,x') \right) = \frac{1}{U_\beta(x)} R\left[f(x)\right] ,
$$

$$
S\left[f(x)\right] = f(0,x') , \qquad S_\beta\left[f(x)\right] = \frac{1}{m_{0\beta}(x')} f(0,x') .
$$

Now we apply this necessary condition of the solvability of the above first order system (F.S.) to the first order system of the transport equations (T.E.). Taking account of lemma 5.2 , we can write this necessary condition in the following form:

$$u^{(j+1)}_{\alpha+2m-2,\beta}(0,x') = -S_\beta\Big[\sum_{k=2}^{m}\sum_{\nu=-1}^{2m-2k} {}^1L_{\nu,k,\beta}(u^{(j+k)}_{\alpha+\nu,\beta}) + {}^3L_{\nu,k,\beta}(g^{(j+k)}_{\alpha+\nu,\beta}) + \sum_{\nu=-1}^{2m-2} {}^1L_{\nu,0,\beta}(u^{(j)}_{\alpha+\nu,\beta}) + {}^3L_{\nu,0,\beta}(g^{(j)}_{\alpha+\nu,\beta}) + \sum_{\nu=-1}^{2m-4} {}^1L_{\nu,1,\beta}(u^{(j+1)}_{\alpha+\nu,\beta}) + {}^3L_{\nu,1,\beta}(g^{(j+1)}_{\alpha+\nu,\beta})\Big] \quad (j \geq 0),$$

$$v^{(j+1)}_{\alpha+2m-2,\beta}(0,x') = -S_\beta\Big[\sum_{k=2}^{m}\sum_{\nu=-1}^{2m-2k} {}^1L_{\nu,k,\beta}(v^{(j+k)}_{\alpha+\nu,\beta}) + {}^3L_{\nu,k,\beta}(h^{(j+k)}_{\alpha+\nu,\beta}) + \sum_{\nu=-1}^{2m-2} {}^1L_{\nu,0,\beta}(v^{(j)}_{\alpha+\nu,\beta}) + {}^3L_{\nu,0,\beta}(h^{(j)}_{\alpha+\nu,\beta}) + \sum_{\nu=-1}^{2m-4} {}^1L_{\nu.1,\beta}(v^{(j+1)}_{\alpha+\nu,\beta}) + {}^3L_{\nu.1,\beta}(h^{(j+1)}_{\alpha+\nu,\beta})\Big] \quad (j \geq 0)$$

Under the necessary condition of the solvability of (F.S.) , (F.S.) is reduced to the next ordinary Fuchsian system of the first order:

$$\begin{cases} \widetilde{L}_\beta g + \widetilde{M}_\beta u = R_\beta\big[D(x)\big] \\ (L_\beta + m_{2\beta})u + \mathcal{O}_\beta \cdot \widetilde{M}'_\beta g = E(x) \end{cases}$$

Taking account of the remarks described above, We can reduce the systems of transport equations (T.E.) to the following form.

$$
(\widetilde{T.E.})
\begin{cases}
(i)
\begin{cases}
\widetilde{L}_\beta(g^{(j)}_{\alpha+2m-1,\beta}) + \widetilde{M}_\beta(u^{(j)}_{\alpha+2m-1,\beta}) \\
= R_\beta\Big[-\sum_{k=1}^{m}\sum_{\nu=-1}^{2m-2k}\Big\{ {}^1L_{\nu,k,\beta}(u^{(j+k)}_{\alpha+\nu,\beta}) + {}^3L_{\nu,k,\beta}(g^{(j+k)}_{\alpha+\nu,\beta})\Big\} \\
\quad -\sum_{\nu=-1}^{2m-2}\Big\{ {}^1L_{\nu,0,\beta}(u^{(j)}_{\alpha+\nu,\beta}) + {}^3L_{\nu,0,\beta}(g^{(j)}_{\alpha+\nu,\beta})\Big\}\Big] , \\
(L_\beta + m_{2\beta})(u^{(j)}_{\alpha+2m-1,\beta}) + \theta_\beta\cdot\widetilde{M}'_\beta(g^{(j)}_{\alpha+2m-1,\beta}) \\
= -\sum_{k=0}^{m}\sum_{\nu=0}^{2m-2k}\Big\{ {}^2L_{\nu,k,\beta}(u^{(j+k)}_{\alpha+\nu,\beta}) + {}^4L_{\nu,k,\beta}(g^{(j+k)}_{\alpha+\nu,\beta})\Big\} \\
\quad -\sum_{\nu=-1}^{2m-2}\Big\{ {}^2L_{\nu,0,\beta}(u^{(j)}_{\alpha+\nu,\beta}) + {}^4L_{\nu,0,\beta}(g^{(j)}_{\alpha+\nu,\beta})\Big\} ,
\end{cases}
\\
(ii)
\begin{cases}
\widetilde{L}_\beta(h^{(j)}_{\alpha+2m-1,\beta}) + \widetilde{M}_\beta(v^{(j)}_{\alpha+2m-1,\beta}) \\
= R_\beta\Big[-\sum_{k=1}^{m}\sum_{\nu=-1}^{2m-2k}\Big\{ {}^1L_{\nu,k,\beta}(v^{(j+k)}_{\alpha+\nu,\beta}) + {}^3L_{\nu,k,\beta}(h^{(j+k)}_{\alpha+\nu,\beta})\Big\} \\
\quad -\sum_{\nu=-1}^{2m-2}\Big\{ {}^1L_{\nu,0,\beta}(v^{(j)}_{\alpha+\nu,\beta}) + {}^3L_{\nu,0,\beta}(h^{(j)}_{\alpha+\nu,\beta})\Big\}\Big] , \\
(L_\beta + m_{2\beta})(v^{(j)}_{\alpha+2m-1,\beta}) + \theta_\beta\cdot\widetilde{M}'_\beta(h^{(j)}_{\alpha+2m-1,\beta}) \\
= -\sum_{k=0}^{m}\sum_{\nu=0}^{2m-2k}\Big\{ {}^2L_{\nu,k,\beta}(v^{(j+k)}_{\alpha+\nu,\beta}) + {}^4L_{\nu,k,\beta}(h^{(j+k)}_{\alpha+\nu,\beta})\Big\} \\
\quad -\sum_{\nu=-1}^{2m-2}\Big\{ {}^2L_{\nu,0,\beta}(v^{(j)}_{\alpha+\nu,\beta}) + {}^4L_{\nu,0,\beta}(h^{(j)}_{\alpha+\nu,\beta})\Big\} .
\end{cases}
\end{cases}
$$

$(\widetilde{T.E.})$ is a linear first order system of $u^{(j)}_{\alpha+2m-1,\beta}$, $g^{(j)}_{\alpha+2m-1,\beta}$, $v^{(j)}_{\alpha+2m-1,\beta}$ and $h^{(j)}_{\alpha+2m-1,\beta}$.

Initial data which we impose on this (T.E.) is obtained in lemma 3.1 for j=0, and for $j \geqq 1$ is obtained ,as,follows;

(I.D.)

$$u^{(j)}_{\alpha+2m-1,\beta}(0,x')=S_\beta\left[\sum_{k=2}^{m}\sum_{\nu=-1}^{2m-2k}\left\{{}^1L_{\nu,k,\beta}\,(u^{(j+k-1)}_{\alpha+1+\nu,\beta})+{}^3L_{\nu,k,\beta}(g^{(j+k-1)}_{\alpha+1+\nu,\beta})\right\}\right.$$

$$+\sum_{\nu=-1}^{2m-2}\left\{{}^1L_{\nu,0,\beta}\,(u^{(j-1)}_{\alpha+1+\nu,\beta})+{}^3L_{\nu.0,\beta}\,(g^{(j-1)}_{\alpha+1+\nu,\beta})\right\}$$

$$\left.+\sum_{\nu=-1}^{2m-4}\left\{{}^1L_{\nu,1,\beta}\,(u^{(j)}_{\alpha+1+\nu,\beta})+{}^3L_{\nu.1,\beta}\,(g^{(j)}_{\alpha+1+\nu,\beta})\right\}\right](j\geqq 1),$$

$$v^{(j)}_{\alpha+2m-1,\beta}(0,x')=S_\beta\left[\sum_{k=2}^{m}\sum_{\nu=-1}^{2m-2}\left\{{}^1L_{\nu,k,\beta}(v^{(j+k-1)}_{\alpha+1+\nu,\beta}+{}^3L_{\nu,k,\beta}(h^{(j+k-1)}_{\alpha+1+\nu,\beta})\right\}\right.$$

$$+\sum_{\nu=-1}^{2m-2}\left\{{}^1L_{\nu,0,\beta}(v^{(j-1)}_{\alpha+1+\nu,\beta})+{}^3L_{\nu,0,\beta}(h^{(j-1)}_{\alpha+1+\nu,\beta})\right\}$$

$$\left.+\sum_{\nu=-1}^{2m-4}\left\{{}^1L_{\nu,1,\beta}(v^{(j)}_{\alpha+1+\nu,\beta})+{}^3L_{\nu,1,\beta}(h^{(j)}_{\alpha+1+\nu,\beta})\right\}\right].$$

We remark above problems take the same form

$$\begin{cases}\widetilde{L}_\beta g+\widetilde{M}_\beta u=R_\beta\left[D(x)\right] \quad , \\ (\widehat{L}_\beta+m_{2\beta})u+\ C_\beta\cdot\widehat{M}'_\beta\ g=E(x)\end{cases}$$

with the initial data $u(0,x')=u_0(x')$, where L_β, $\widehat{L}_\beta$, $\widetilde{M}_\beta$ and $\widetilde{M}'_\beta$ have holomorphic coefficients and $m_{2\beta}$, $D(x)$ and $E(x)$, $u_0(x')$

are holomorphic in a neighbourhood of $0 \in C^{n+1}$. It is known that for this Cauchy problem there exist unique holomorphic solutions $u(x)$ and $g(x)$ in a neighbourhood of $0 \in C^{n+1}$.

By this fact, holomorphic coefficients $u_{\alpha,\beta}^{(q)}$, $g_{\alpha,\beta}^{(q)}$, $v_{\alpha,\beta}^{(q)}$ and $h_{\alpha,\beta}^{(q)}$ can be determined inductively.

First, we suppose all $u_{\gamma,\beta}^{(p)}$, $g_{\gamma,\beta}^{(p)}$, $v_{\gamma,\beta}^{(p)}$ and $h_{\gamma,\beta}^{(p)}$ ($0 \leq p$, $\beta=1,2\cdots,m$ for $\gamma \leq \alpha+2m-2$, and $0 \leq p \leq j-1$, $\beta=1\cdots m$ for $\gamma=\alpha+2m-1$,) are determined and then the right hand side of (T.E.) are known.

For $j=0$, using lemma 3.1 and for $j \geq 1$, using (I.D.), we can solve the Cauchy problem for (T.E.) and then we can determine $u_{\alpha+2m-1,\beta}^{(j)}$, $g_{\alpha+2m-1,\beta}^{(j)}$, $v_{\alpha+2m-1,\beta}^{(j)}$ and $h_{\alpha+2m-1,\beta}^{(j)}$ inductively.

We shall prove that these coefficients $u_{\alpha,\beta}^{(q)}$, $g_{\alpha,\beta}^{(q)}$, $v_{\alpha,\beta}^{(q)}$ and $h_{\alpha,\beta}^{(q)}$ have a common existence domain and suitable estimates.

§.7 New coordinates and ${}^{h}L_{\nu,k,\beta}$.

To make (T.E.) easier in the treatment in the estimation of the coefficients, we introduce the new coordinates $y_\beta=(y_{0,\beta},\cdots,y_{n,\beta})$ $(\beta=1,\cdots,m)$, as follows.

We set $\begin{cases} y_{0,\beta}=x_0 \\ y_{i,\beta}=\psi_{i,\beta}(x) \qquad (i=1,\cdots,n) \end{cases}$

, where $\psi_{i,\beta}(x)$ are defined as the solutions of the Cauchy problem:

$$\sum_{j=1}^{n} P^{(j)}(e_\beta)D_j\psi_{i,\beta}(x)=0$$

with the initial data $\psi_{i,\beta}(0,x')=x_1$.

Considering the transformation of the coordinates x into the new coordinates y_β , we have

$$\begin{cases} D_0=D_{0,\beta}+\sum_{i=1}^{n}\psi_{i,\beta_{x_0}}(x)D_{i,\beta} \\ D_\gamma=\sum_{i=1}^{n}\psi_{i,\beta_{x_\gamma}}(x)D_{i,\beta} \qquad (\gamma=1,\cdots,n) \end{cases}$$

where $D_{i,\beta}=\dfrac{\partial}{\partial y_{i,\beta}}$ and $\psi_{1,\beta_x}(0,x')=\rho_{\beta_x}(0,x')$.

We are to see some properties of ${}^{h}L_{\nu,k,\beta}$ in terms of the new coordinates y_β . L_β and M_β are expressed as follows:

$$\begin{cases} L_\beta = a_\beta[y_\beta]\, D_{0,\beta} + y_{0,\beta}\left\{ \sum_{i=0}^{n} a_{\beta,i}[y_\beta]\, D_{i,\beta} \right\} + c_\beta[y_\beta] & , \\ M_\beta = b_\beta[y_\beta]\, y_{0,\beta} D_{0,\beta} + y_{0,\beta}^{2}\left\{ \sum_{i=0}^{n} b_{\beta,i}[y_\beta]\, D_{i,\beta} + c_\beta'[y_\beta] \right\} & , \end{cases}$$

where a_β , $a_{\beta,i}$, b_β , $b_{\beta,i}$, c_β and c_β' are holomorphic functions of y_β in a neighourhood of the origin. And so we have the next lemma with respect to the transport operators ${}^{h}L_{2m-1,0,\beta}$.

Lemma 7.1 ${}^{h}L_{2m-1,0,\beta}$ are expressed, in termes of the new coordinates y_β , as follows.

(i) $$ {}^{1}L_{2m-1,0,\beta} = y_{0,\beta} b_\beta[y_\beta]\, D_{0,\beta} + y_{0,\beta}^{2} \sum_{i=0}^{n} b_{\beta,i}[y_\beta]\, D_{i,\beta} + y_{0,\beta} c_{1,\beta}[y_\beta] \quad ,$$

(ii) $$ {}^{2}L_{2m-1,0,\beta} = a_\beta[y_\beta]\, D_{0,\beta} + y_{0,\beta} \sum_{i=0}^{n} a_{\beta,i}[y_\beta]\, D_{i,\beta} + c_{2,\beta}[y_\beta] \quad ,$$

(iii) $$ {}^{3}L_{2m-1,0,\beta} = \mathcal{C}_\beta[y_\beta]\left[a_\beta'[y_\beta]\left\{ y_{0,\beta} D_{0,\beta} + 1 \right\} + y_{0,\beta}^{2}\left\{ \sum_{i=0}^{n} a_{\beta,i}[y_\beta] D_{i,\beta} + c_{3,\beta}[y_\beta] \right\} \right] \quad ,$$

(iv) $$ {}^{4}L_{2m-1,0,\beta} = y_{0,\beta} b_\beta[y_\beta]\, D_{0,\beta} + y_{0,\beta}^{2} \sum_{i=0}^{n} b_{\beta,i}[y_\beta]\, D_{i,\beta} + y_{0,\beta} c_{4,\beta}[y_\beta]$$

where $a_\beta[y_\beta] = 2(P^{(0)}(e_\beta))^2 \sigma_\beta(0,x') \neq 0$,

$a_\beta'[y_\beta] = 2m_{0\beta}(x') \neq 0$

$b_\beta[y_\beta] = P_\beta(x')P^{(0)}(e_\beta) \neq 0$

Among other operators ${}^{h}L_{\nu,k,\beta}$, we see ${}^{h}L_{\nu,0,\beta}$ and ${}^{3}L_{\nu,1,\beta}$

Taking account of (F.5) and lemma 5.3 , we have,

Lemma 7.2 Exchanging x for y_β , we get the following representations of ${}^{h}\overset{\circ}{L}_{\nu,0,\beta}$ (h = 1,2,3,4) , ${}^{3}L_{\nu,0,\beta}$ and ${}^{3}\overset{\circ}{L}_{\nu,1,\beta}$.

(i) The principal part of ${}^{1}\overset{\circ}{L}_{\nu,0,\beta}[y_\beta, D_{y_\beta}]$,equal to ${}^{4}\overset{\circ}{L}_{\nu,0,\beta}[y_\beta, D_{y_\beta}]$, can be expressed in the form :

$$H[y_\beta, D_{y_\beta}]\, D_{0,\beta}^{2} + y_{0,\beta} K[y_\beta, D_{y_\beta}]\, D_{0,\beta} + y_{0,\beta}^{2} J[y_\beta, D_{y_\beta}] \quad .$$

(ii) The principal part of ${}^{2}\overset{\circ}{L}_{\nu,0,\beta}[y_\beta, D_{y_\beta}]$ and the principal part of ${}^{3}\overset{\circ}{L}_{\nu,1,\beta}[y_\beta, D_{y_\beta}]$ can be expressed in the form:

$$y_{0,\beta}^{2} K[y_\beta, D_{y_\beta}]\, D_{0,\beta} + y_{0,\beta} J[y_\beta, D_{y_\beta}] \quad .$$

(iii) The principal part of ${}^{3}\overset{\circ}{L}_{\nu,0,\beta}[y_\beta, D_{y_\beta}]$ can be expressed in the form :

$$y_{0,\beta}^{2} K[y_\beta, D_{y_\beta}]\, D_{0,\beta} + y_{0,\beta}^{3} J[y_\beta, D_{y_\beta}] \quad .$$

(iv) $${}^{3}L_{\nu,0,\beta}[y_\beta, D_{y_\beta}] \equiv I[y_\beta, D_{y_\beta}]\, D_{0,\beta} + (\text{operators of order } 2m-\nu-2) \quad \text{mod. } x_0 \quad ,$$

where $I[y_\beta, D_{y_\beta}]$ is a linear partial differential operator of order $2m-\nu-2$.

§.8 Convergence of the formal solution.

After the construction of the formal solution which has been done in §.3 and §.7 , it remains for us to verify the convergence of the formal solution. To do so , we studied how the holomorphic coefficients were determined, that is , in §.6 they were determined succesively by solving the Cauchy problem for the systems of the transport equations $(\widetilde{T.E.})$ with the initial data by (I.D.) or lemma 3.1 . Moreover we study the systems of the transport equations $(\widetilde{T.E.})$, especially, transport operators precisely and other operators appearing in right hand sides of the transport equations, in §.5 and §.7. In this section, we are to prove the convergence of the formal solution by the majoration method . To do so, we begin it with the introduction of a family of the scale functions $\{\phi_j(t,s)\}$.

We define $\phi_j(t,s)$ as follows.

$$\phi_j(t,s) \equiv \partial_s^j \phi(t,s)$$

,where $\phi(t,s) = (\sqrt{2(R-s)} - \rho t)^{-1} \qquad (\rho > 1)$.

We put $R[f(t,s)] = t^{-1}(f(t,s)-f(0,s))$. The following proposition can be easily verified.

Proposition 6.1 $\{\phi_j(t,s)\}$ have following properties.

$$(1) \qquad \partial_t\phi_j \gg \rho t\phi_{j+1} \quad .$$

$$(2) \qquad \partial_t^2\phi_j \gg \rho^2\phi_{j+1} \quad .$$

$$(3) \qquad \partial_t^2\phi_j \gg \rho^2 t^2\phi_{j+2} \quad .$$

$$(4) \qquad (t\partial_t+1)\phi_{j+1} \gg \partial_t\phi_j \ , \ R[\partial_t\phi_j] \ , \ \partial_t^2\phi_j \quad .$$

$$(5) \qquad 8\rho(t\ {}_t+1)\phi_{j+2} \gg \partial_t^3\phi_j \ , \ R[\partial_t^3\phi_j] \quad .$$

$$(6) \qquad 2\phi_{j+1}(0,s) \gg \partial_t^2\phi_j(0,s) \quad .$$

$$(7) \qquad 4\partial_t^2\phi_{j+1}(0,s) \gg \partial_t^4\phi_j(0,s) \quad .$$

$$(8) \qquad \frac{1}{(R'-s)(R''-t)}\partial_t^k\phi_j \ll \frac{1}{(R'-R)(R''-R)}\partial_t^k\phi_j \ \begin{pmatrix} j\geq 1, \ R\leq 1, \\ R'', R' > R \end{pmatrix} \quad .$$

The majoration method applied to the proof of the convergence of the formal solution is based on the next proposition.

Proposition 6.2 For the Cauchy problem

$$\left\{ \begin{aligned} &(x_0D_0+1)g + \left(\sum_{i=0}^{n} x_0^2\alpha_i D_i + x_0\gamma_1\right)g + \\ &\qquad\qquad (\gamma_0 D_0 + \sum_{i=0}^{n} x_0\beta_i D_i + \gamma_2)u \quad = T(x) \end{aligned} \right.$$

$$\left\{\begin{array}{l} D_0 u + \left(\sum_{i=0}^{n} x_0 \delta_i D_i + \gamma_3 \right) u + \\ \qquad x_0 \left(\gamma_0 D_0 + \sum_{i=0}^{n} x_0 \beta_i D_i + \gamma_4 \right) g \quad = \quad T(x) \end{array}\right. ,$$

with the initial data $u(0,x') = u_0(x')$,

where α_i , β_i , γ_i , δ_i , S, T and u_0 are holomorphic functions in a neighbourhood of the origin , there exist unique holomorphic solutions $u(x)$ and $g(x)$ in a neighbourhood of the origin.

Moreover, assuming $\alpha_i \ll \tilde{\alpha}_i$, $\beta_i \ll \tilde{\beta}_i$, $\gamma_i \ll \tilde{\gamma}_i$, $\delta_i \ll \tilde{\delta}_i$, $S \ll \tilde{S}$, $T \ll \tilde{T}$ and $u_0 \ll \tilde{u}_0$, we can verify that $u(x) \ll \tilde{u}(x)$ and $g(x) \ll \tilde{g}(x)$, if $\tilde{u}(x)$ and $\tilde{g}(x)$ satisfy

$$(x_0 D_0 + 1)\tilde{g} \gg \left(\sum_{i=0}^{n} x_0^2 \tilde{\alpha}_i + x_0 \tilde{\gamma}_1 \right) \tilde{g} + \left(\tilde{\gamma}_0 D_0 + \sum_{i=0}^{n} x_0 \tilde{\beta}_i D_i + \tilde{\gamma}_2 \right) \tilde{u} + \tilde{S} ,$$

$$D_0 \tilde{u} \gg \left(\sum_{i=0}^{n} x_0 \tilde{\delta}_i + \tilde{\gamma}_3 \right) \tilde{u} + x_0 \left(\tilde{\gamma}_0 D_0 + \sum_{i=0}^{n} x_0 \tilde{\beta}_i D_i + \tilde{\gamma}_4 \right) \tilde{g} + \tilde{T} ,$$

$$\tilde{u}(0,x') \gg \tilde{u}_0(x') ,$$

(for the proof see [10]).

From these propositions and lemmata obtained in the preceeding sections, we have reached the following proposition.

Proposition6.3 There exist positive constants A,B,C,D,E,F, K,L,R and ρ independent of α and j , such that

(1) $u^{(j)}_{\alpha,\beta} \ll A\frac{K^{\alpha}}{L^{j}}\partial_t \phi_{\alpha+j+l+1}(y_{0,\beta};y'^{\#}_{\beta})$,

(2) $g^{(j)}_{\alpha,\beta} \ll B\frac{K^{\alpha+1}}{L^{j}} \phi_{\alpha+j+l+2}(y_{0,\beta};y'^{\#}_{\beta})$,

(3) $v^{(j)}_{\alpha,\beta} \ll C\frac{K^{\alpha}}{L^{j}}\partial_t \phi_{\alpha+j+l+2}(y_{0,\beta},y'^{\#}_{\beta})$,

(4) $h^{(j)}_{\alpha,\beta} \ll D\frac{K^{\alpha+1}}{L^{j}} \phi_{\alpha+j+l+3}(y_{0,\beta},y'^{\#}_{\beta})$,

(5) $(u^{(0)}_{\alpha,\beta} + h^{(0)}_{\alpha-1,\beta})(0,x') \ll EK^{\alpha}\partial_t \phi_{\alpha+j+l+1}(0,x'^{\#})$,

(6) $u^{(0)}_{\alpha,\beta}(0,x') \ll FK^{\alpha}\partial_t \phi_{\alpha+j+l+1}(0,x'^{\#})$,

where $y'^{\#}_{\beta} = \sum_{i=1}^{n} y_{i,\beta}$ and $x'^{\#} = \sum_{i=1}^{n} x_i$.

From this proposition, we know that holomorphic coefficients $u^{(j)}_{\alpha,\beta}$, $g^{(j)}_{\alpha,\beta}$, $v^{(j)}_{\alpha,\beta}$ and $h^{(j)}_{\alpha,\beta}$ have a common existence domain that is a neighbourhood of $0 \in C^{n+1}$ and the estimates $|u^{(j)}_{\alpha,\beta}|$ $|g^{(j)}_{\alpha,\beta}|$, $|v^{(j)}_{\alpha,\beta}|$, $|h^{(j)}_{\alpha,\beta}| < C|\Gamma(\alpha+j+1)|\frac{K^{\alpha}}{L^{j}}$ in this common existence domain, where C,L and K are positive constants independent of α and j . On the other hand, in Appendix we have the estimates $\left|X^{(j)}_{\alpha,\beta}\right|$, $\left|Y^{(j)}_{\alpha,\beta}\right| < C_K \left|\frac{\Psi(\alpha+j+1)}{\Gamma(\alpha+j+1)}\right| r^{\alpha+j}(\log r)^j C^{\alpha+j}$ on any compact set $\underset{\approx}{K}$ in the universal covering space $\widetilde{D_r \setminus \underset{\sim}{K}}$ over $D_r \setminus \underset{\sim}{K}$, where $\alpha > 0$, and $C_{\underset{\approx}{K}}$ is a constant which depends only on $\underset{\approx}{K}$. Thus choosing $r > 0$ sufficiently small, we can prove the convergence of the formal solution on $\widetilde{D_r \setminus \underset{\sim}{K}}$. Uniqueness of the solution is due to the Cauchy- Kobalevskaya theorem. We remark $u(x)$ does not ramify on $x_0 = 0 \wedge x_1 \neq 0$.

Correction. In §.6 of the author's preceeding paper [11], we must replace $\phi_\alpha(z,\zeta,y)$ by the new $\phi_\alpha(z,\zeta,y) = \phi_\alpha(z,\zeta+y,0)$, (where ϕ_α of the right hand side of this identity is one defined in §.6 of [11]) . With this replacement of the old ϕ_α by the new ϕ_α , we have only to replace $[R'-(2/3)][R''-\rho z-y]$ by $[R'-(2/3)(\zeta+y)][R''-\rho z]$ in (5) of Proposition6.1 and $M(R'-(2/3)y_{1,\beta})^{-1}_{,}(R''-y_{0,\beta}-\sum_{\nu=2}^{n} y_{\nu,\beta})^{-1}$ by $M(R'-(2/3)\sum_{\nu=1}^{n} y_{\nu,\beta})^{-1}$ $\cdot(R''-y_{0,\beta})^{-1}$, taking account of the fact that the principal part of ${}^{3^\circ}L_{\nu,\beta}$ is expressed in the form $y_{0,\beta}K[y_\beta, D_{y_\beta}]D_{y_{0,\beta}}$ +

$y^2_{0,\beta} J[y_\beta, D_{y_\beta}]$ and that the principal part of ${}^h\overset{\circ}{L}_{\nu,\beta}$ (h=1,2,4) are expressed in the form $K[y_\beta, D_{y_\beta}]D_{y_0,} + y_{0,\beta} J[y_\beta, D_{y_\beta}]$, (for the proof of this fact , see the proof of lemma 7.2 of this paper).

§.9 Proof.

Proof of Lemma 1.1 We differentiate by c both sides of $P_c(U_{\alpha+q})=0$ q times and then we get the first relation inductively

We prove other relations in the similar way.

Proof of Proposition 4.1 Consider the Cauchy problem

$$\overset{\circ}{L}(x,\varphi_x)=P(x,\varphi_x)^2 - x_0^2 Q(x,\varphi_x)=0$$

with the initial data $\varphi(0,x')=x_1$

From this, we have the new Cauchy problem

$$(*) \quad P(x,\varphi_x)= \pm x_0\, Q(x,\varphi_x)$$

with the initial data $\varphi(0,x')=x_1$

Taking account of (P)(ii),(iii) and (Q)(ii), Cauchy-Kowalevskaya theorem and implicit function theorem guarantee that this Cauchy problem(*) has 2m solutions $\varphi_\beta^{\pm}(x)$ which are holomorphic .

Differentiating the equation (*) by x_0 and restricting it on the initial surface, we have $\varphi^{\pm}_{\beta x_0}(0,x')=\lambda_\beta(x')$ and $\varphi^{\pm}_{\beta x_0 x_0}(0,x')$ $=\left[P^{(0)}(e_\beta)\right]^{-1}\left\{\pm\sqrt{Q(e_\beta)}-P_{x_0}(e_\beta)-\sum_{i=1}^{n}\lambda_{\beta x_i}(x')P^{(i)}(e_\beta)\right\}$. Then we have $\rho_\beta(x)=\frac{1}{2}(\varphi_\beta^+(x)+\varphi_\beta^-(x))=x_1+x_0\lambda_\beta(x')+x_0^2\tau_\beta(x)$ and

$\theta_\beta(x)=(\varphi_\beta^+(x)-\varphi_\beta^-(x))^{1/2}=x_0\sigma_\beta(x)$, where $\tau_\beta(0,x')=-\left[P^{(0)}(e_\beta)\right]^{-1}$
$\left\{P_{x_0}(e_\beta)+\sum_{i=1}^{n}\lambda_{\beta x_i}(x')P^{(i)}(e_\beta)\right\}$ and $\sigma_\beta(0,x')=\left[\frac{\sqrt{Q(e_\beta)}}{P^{(0)}(e_\beta)}\right]^{1/2}$

Proof of Lemma 5.1

(i) $S[L_\beta]=S\left[\sum_{j=0}^{n}\overset{o}{L}_1{}^{(j)}(0,x',\theta_{\beta x},\rho_{\beta x})D_j\right]=\sum_{i,j=0}^{n}\theta_{\beta x_i}\overset{o}{L}{}^{(i,j)}(e_\beta)D_j\Big|_{x_0=0}$

$=\sigma_\beta(0,x')\sum_{j=0}^{n}\overset{o}{L}{}^{(0,j)}(e_\beta)D_j$ and $\overset{o}{L}{}^{(0,j)}(e_\beta)=2P^{(0)}(e_\beta)P^{(j)}(e_\beta)$

(ii) $$\begin{aligned} M_\beta &= \sum_{j=0}^{n}\left[\sum_{i=0}^{m-1}\overset{o}{L}{}^{(j)}_{2i}(x,\theta_{\beta x},\rho_{\beta x})\theta_\beta^{2i}\right]D_j \\ &\equiv \sum_{j=0}^{n}\overset{o}{L}{}^{(j)}_{0}(x,\theta_{\beta x},\rho_{\beta x})D_j && \text{mod. } \theta_\beta{}^2 \\ &\equiv \sum_{j=0}^{n}\overset{o}{L}{}^{(j)}(x,\rho_{\beta x})D_j && \text{mod. } \theta_\beta{}^2 \\ &\equiv \sum_{j=0}^{n}P(x,\rho_{\beta x})P^{(j)}(x,\rho_{\beta x})D_j && \text{mod. } \theta_\beta{}^2 \end{aligned}$$

On the other hand by $S\left[P(x,\rho_{\beta x})\right]=S\left[P(x,\varphi^{\pm}_{\beta x})\right]=0$, we can put

$P(x,\rho_{\beta x})=x_0P_\beta'(x)$. We proved (ii).

(iii) Note $S\left[\frac{1}{\theta_\beta}{}^3L_{2m-1,0,\beta}\right]=S\left[\frac{1}{\theta_\beta}(N'_{c,\beta}+R'_{3,\beta})\right]$

$$\begin{aligned} N'_{c,\beta} &= \sum_{i=0}^{m}\overset{o}{L}_{2i-1}(x,\theta_{\beta x},\rho_{\beta x})c_\beta\cdot i\,\theta_\beta^{2i-2} \\ &\equiv c\,\overset{o}{L}_1(x,\theta_{\beta x},\rho_{\beta x}) && \text{mod. } \theta_\beta{}^2 \end{aligned}$$

On the other hand from the eikonal equation ${}^2L_{2m,0,\beta}=$

$= \overset{\circ}{L}_1(x,\theta_{\beta x},\rho_{\beta x}) + \sum_{i=1}^{m-1} \overset{\circ}{L}_{2i+1}(x,\theta_{\beta x},\rho_{\beta x})\theta_\beta^{2i} = 0$, we get

$\overset{\circ}{L}_1(x,\theta_{\beta x},\rho_{\beta x}) \equiv 0 \mod. \theta_\beta^2$. $R'_{3,\beta} = \sum_{i=0}^{m} \overset{\circ}{L}_{2i}(x,\theta_{\beta x},\rho_{\beta x}) 2i^2 \theta_\beta^{2i-1}$

$\equiv \overset{\circ}{L}_2(x,\theta_{\beta x},\rho_{\beta x})\ 2\theta_\beta \mod. \theta_\beta^2$ and $\overset{\circ}{L}_2(x,\theta_{\beta x},\rho_{\beta x}) =$

$\frac{1}{2}\sum_{i,j=0}^{n} \theta_{\beta x_i}\theta_{\beta x_j} \overset{\circ}{L}^{(i,j)}(x,\rho_{\beta x})$ lead to (iii).

(iv) $S\left[{}^1L_{2m-1,0,\beta}\right] = S\left[{}^4L_{2m-1,0,\beta}\right]$

$= S\left[\frac{1}{2}\sum_{i,j=0}^{n}\{\rho_{\beta x_i x_j} \overset{\circ}{L}{}_0^{(i,j)}(x,\theta_{\beta x},\rho_{\beta x})\} + c_\beta \overset{\circ}{L}_2(x,\theta_{\beta x},\rho_{\beta x}) + \overset{\circ}{L}_{s,0}(x,\theta_{\beta x}\right.$

$\left.,\rho_{\beta x})\right] = S\left[\frac{1}{2}\sum_{i,j=0}^{n}(\rho_{\beta x_i x_j} + c_\beta\,\theta_{\beta x_i}\theta_{\beta x_j})\overset{\circ}{L}^{(i,j)}(x,\rho_{\beta x}) + \overset{\circ}{L}_s(x,\rho_{\beta x})\right]$

$= \frac{1}{2}\{\gamma_\beta(0,x')P^{(0,0)}(e_\beta) + 2\sum_{j=1}^{n}\lambda_{\beta x_j}(x')P^{(0,j)}(e_\beta) + c_\beta(\sigma_\beta(0,x'))^2 P^{(0,0)}(e_\beta)$

$+ L_s(e_\beta) = 0$. This is (L_s)(i) itself.

Proof of Lemma 5.2

$S\left[{}^1L_{2m-2,1,\beta}\right] = S\left[\overset{\circ}{L}_2(x,\theta_{\beta x},\rho_{\beta x})\right] = S\left[\frac{1}{2}\sum_{i,j=0}^{n}\theta_{\beta x_i}\theta_{\beta x_j}\overset{\circ}{L}^{(i,j)}(x,\rho_{\beta x})\right]$

$= m_{0,\beta}(x') \neq 0$.

Proof of Lemma5.3 (F.5) and (F.2) lead to Lemma 5.3 .

Proof of Lemma 7.2

(i) $\overset{\circ}{{}^1L}_{\nu,0} = \overset{\circ}{{}^4L}_{\nu,0} \equiv \sum_{|\alpha|=2m-\nu} \overset{\circ}{L}{}_0^{(\alpha)}(x,\theta_x,\rho_x)\frac{D^\alpha}{\alpha!} \mod. x_0^2$

$\equiv \sum_{|\alpha|=2m-\nu} \overset{\circ}{L}^{(\alpha)}(x,\rho_x)\frac{D^\alpha}{\alpha!} \mod. x_0^2$.

On the other hand, by Leibniz's formula

$$L^{(\alpha)}(x,\xi) \equiv \sum_{|\beta+\gamma|=2m-\nu} \frac{\alpha!}{\beta!\gamma!} P^{(\beta)}(x,\xi)P^{(\gamma)}(x,\xi) \quad \text{mod. } x_0^2$$

we get ${}^1L^{0}_{\nu,0} \equiv$ the principal part of

$$\sum_{|\beta+\gamma|=2m-\nu} \left\{P^{(\beta)}(x,\xi)\frac{D^\beta}{\beta!}\right\}\left\{P^{(\gamma)}(x,\xi)\frac{D^\gamma}{\gamma!}\right\} \qquad \text{mod. } x_0^2$$

$\equiv$ the principal part of

$$\sum_{k=0}^{2m-k} \left\{ \sum_{|\beta|=k} P^{(\beta)}(x,\xi)\frac{D^\beta}{\beta!}\right\}\left\{\sum_{|\gamma|=2m-\nu-k} P^{(\gamma)}(x,\xi)\frac{D^\gamma}{\gamma!}\right\} \text{mod. } x_0^2$$

Therefore it suffices to prove that the principal part of the

operator obtained by exchanging x for y_β from

$\sum_{|\alpha|=k} P^{(\alpha)}(x,\xi)\frac{D^\alpha}{\alpha!}$ $\quad (\xi = {}^t\!f'_{\beta\, x})$ is expressed in the form $K(y_\beta, D_{y_\beta})D_{y_{0,\beta}}$ $+ y_{0,\beta}J(y_\beta, D_{y_\beta})$. On the other hand, $P(x,\xi)=R(x,\xi)r(x,\xi)$

$(r(x,\xi)=\xi_0-\lambda_\beta(x,\xi'))$ leads to

$P^{(\alpha)}(x,\xi) = \sum_{\gamma+\delta=\alpha} \frac{\alpha!}{\gamma!\delta!} R^{(\delta)}(x,\xi)r^{(\gamma)}(x,\xi)$. So, we get

$$\sum_{|\alpha|=k} \frac{P^{(\alpha)}(x,\xi)}{\alpha!}D^\alpha = \text{the principal part of}$$

$$\left\{\sum_{|\delta|=j} R^{(\delta)}(x,\xi)\frac{D^\delta}{\delta!}\right\}\left\{\sum_{|\gamma|=k-j} r^{(\gamma)}(x,\xi)\frac{D^\gamma}{\gamma!}\right\}$$

And so, it is sufficient for us to prove that the principal

part of the operator obtained by exchanging x for y_β from

$\sum_{|\alpha|=k} r^{(\alpha)}(x,\xi)\frac{D^{\alpha}}{\alpha!}$ ($k=0,\cdots,m$, $\xi = \rho_{\beta x}$) is expressed in the form $K(y_\beta, D_{y_\beta})D_{y_{0,\beta}} + y_{0,\beta}J(y_\beta, D_{y_\beta})$. On the other hand , $r(x,\rho_{\beta x}) = r(e_\beta)$ mod. x_0 .

For k=0, by $r(e_\beta)=0$, we get $r(x,\rho_{\beta x}) = y_{0,\beta}J(y_\beta)$.

For k=1, we get $\sum_{i=0}^{n} r^{(i)}(e_\beta)D_i = D_{y_{0,\beta}}$.

For $k \geq 2$, Euler's identity with respect to homogeneous polynomials leads to $r^{(\alpha)}(e_\beta) = 0$.

We proved (i) and then we can prove (ii),(iii) and (iv) in the similar way.

Proof of Proposition 6.3 We use M as a suitable positive constant in this proof. Let all coefficients of ${}^{h}L_{\nu,k,\beta}$ $\ll M(R'-y'^{\#}_{\beta})^{-1}(R''-\rho\, y_{0,\beta})$. We prove this proposition by induction of α and j . Assume that these estimates are valid for $\alpha = -1+1, \cdots, \alpha+2m-2$ and $j \geq 0$. First we treat (5).

Lemma3.1 leads to $S\left[u^{(0)}_{\alpha+2m-1,\beta} + h^{(0)}_{\alpha+2m-2,\beta}\right] =$

$$\sum_{\mu=1}^{2m-2} {}^{1}N^{(0)}_{\mu,\beta}\left(u^{(0)}_{\alpha+2m-1-\mu,\beta} + h^{(0)}_{\alpha+2m-2-\mu,\beta}\right) + (\text{ rest }) \Big|_{x_0=0}$$

Making use of the fact that $D_0^{\mu} = S_{\beta}(y_{\beta}, D_{y_{\beta}})D_{y_{0,\beta}} + T_{\beta}(y_{\beta}, D'_{y_{\beta}})$ where S_{β} is a linear partial differential operator of order $\mu - 1$ in $D_{y_{\beta}}$ and T_{β} is a linear partial differential operator of order μ in $D'_{y_{\beta}} = (D_{y_{1,\beta}}, \cdots, D_{y_{n,\beta}})$, we have

$$S\left[D_0^{\mu}(u^{(0)}_{\alpha+2m-1-\mu,\beta} + h^{(0)}_{\alpha+2m-2-\mu,\beta})\right] =$$

$$S(0, y'_{\beta}, D_{y_{\beta}})D_{y_{0,\beta}}(u^{(0)}_{\alpha+2m-1-\mu,\beta} + h^{(0)}_{\alpha+2m-2-\mu,\beta})(y_{\beta}) +$$

$$T_{\beta}(y_{\beta}, D'_{y_{\beta}})(u^{(0)}_{\alpha+2m-1-\mu,\beta} + h^{(0)}_{\alpha+2m-2-\mu,\beta})(y_{\beta})\Big|_{x_0=0} .$$

To the former part of the right hand side of this identity, we apply the estimates of (1),(4) of the assumption of the induction, and to the latter part of the right hand side of this identity , we apply the estimates of (5) of the assumption of the induction. Then we have

$$S\left[u^{(0)}_{\alpha+2m-1,\beta} + h^{(0)}_{\alpha+2m-2,\mu}\right] \ll K^{\alpha+2m-3}M(KD+A+B+C+D)\partial_t \phi_{\alpha+2m+1} .$$

For (5), it suffices that the inequality

(i) $\quad EK^2 > MKD+(A+B+C+D)M$

is valid.

Secondly we treat (6). We restrict the transport equation $\tilde{L}_{\beta}(h^{(0)}_{\alpha+2m-2,\beta}) = -\tilde{M}_{\beta}(v^{(0)}_{\alpha+2m-2,\beta}) + R_{\beta}[\cdots]$ on the initial surface , and then we have

$$h^{(0)}_{\alpha+2m-2,\beta}(0,x') \ll K^{\alpha+2m-2}\Big\{C\partial_t^2 \phi_{\alpha+2m+1} + R[\phi_{\alpha+2m+1}]K + R\Big[\partial_t \phi_{\alpha+2m+1-1}\Big]\Big\}\Big|_{x_0=0} .$$

On the other hand , $S\big[u^{(0)}_{\alpha+2m-1,\beta}\big] = S\big[u^{(0)}_{\alpha+2m-1,\beta} + h^{(0)}_{\alpha+2m-2,\beta}\big] - S\big[h^{(0)}_{\alpha+2m-2,\beta}\big]$. To the former part of the right hand side of this identity, we apply the estimates of (5) and to the latter part of the right hand side of this identity , we apply the estimates obtained above. For (6), it suffices that the inequality

(ii) $\quad F > (E+D)M$

is valid.

Thirdly for (1) and (2) , by (T.E.) it is known, it suffices that the following system of the inequalities (iii) is valid.

(iii) $\quad \Big\{ \rho B(1-\frac{M}{\rho})K > M(A+B) + \frac{M}{LK}(A+B)$

$$\left. A\left(1-\frac{M}{\rho}\right) > M\frac{K}{\rho}B + \frac{M}{LK}(A+B) \right.$$

Lastly we treat (3) and (4) lead to

$$v^{(0)}_{\alpha+2m-1,\beta}(0,x') \ll EK^{\alpha+2m}\partial_t^{l}\phi_{\alpha+2m-1+2+1}(0,x'^{\#})$$

For (3) and (4), it suffices that the following system of the inequalities (iv) is valid .

$$\text{(iv)} \qquad \begin{cases} \rho D\left(1-\frac{M}{\rho}\right)K > M(C+D) + \frac{M}{LK}(C+D) \\ C\left(1-\frac{M}{\rho}\right) > M\frac{K}{\rho}D + \frac{M}{LK}(C+D) \\ C > E \end{cases}$$

For $j \geq 1$, we treat them in the similar way , but lemma 3.1 is to be replaced by (I.D.) . So the following system of inequalities must be satisfied.

$$\text{(v)} \qquad \begin{cases} A > M(K^{-2}A+K^{-1}B+L^{-1}B) \\ C > M(K^{-2}C+K^{-1}D+L^{-1}D) \end{cases}$$

Our problem is reduced to the problem of the existence of positive constants A,B,C,D,E,F,K,L,R and ρ which satisfy the system of inequalities (i),(ii),(iii),(iv) and(v) . In fact, In fact, making ρ, K and L sufficiently large and R is sufficiently small, we can make these constant satisfy the system of inequalities (i)(ii)(iii)(iv) and (v).

Appendix

In this appendix, we study the multi-valuedness or the singularity of the auxiliary functions U_α, V_α, $U_\alpha^{(j)}$, $V_\alpha^{(j)}$, X_α, Y_α, $X_\alpha^{(j)}$ and $Y_\alpha^{(j)}$ which are the analytic functions of θ, ρ, α and c. We use, as independent variables, $(t,x) \in C^2$ instead of $(\theta,\rho) \in C^2$ in this appendix, and treat these auxiliary functions in the more general form.

To be precise, we put

$$P_c = \partial_t^2 - t^{2b}\partial_x^2 - ct^{b-1}\partial_x , \quad (b>0 \text{ is an integer})$$

Newly we introduce U_α and V_α as the solutions of the following Cauchy problems respectively :

$$P_c U_\alpha = (\partial_t^2 - t^{2b}\partial_x^2 - ct^{b-1}\partial_x)U_\alpha = 0$$

with the initial data $\begin{cases} U_\alpha(0,x) = f_\alpha(x) \\ U_{\alpha t}(0,x) = 0 \end{cases}$

$$P_c V_\alpha = (\partial_t^2 - t^{2b}\partial_x^2 - ct^{b-1}\partial_x)V_\alpha = 0$$

with the initial data $\{ V_\alpha(0,x) = 0$

$$\left. V_{\alpha t}(0,x) = f_\alpha(x) \right.$$

We denote $\dfrac{A}{2(b+1)}$ by A^* .

The following explicit representation are known, (see [1]).

$$\begin{cases} U_\alpha(t,x) = \dfrac{\xi^\alpha}{\Gamma(\alpha+1)}F(-\alpha, b^*+c^*, 2b^*, z) \\ \qquad\qquad = \dfrac{\eta^\alpha}{\Gamma(\alpha+1)}F(-\alpha, b^*-c^*, 2b^*, \zeta) \quad , \\ V_\alpha(t,x) = \dfrac{\xi^\alpha t}{\Gamma(\alpha+1)}F(-\alpha, c^*-b^*+1, 1+2^*, z) \\ \qquad\qquad = \dfrac{\eta^\alpha t}{\Gamma(\alpha+1)}F(-\alpha, 1-b^*-c^*, 1+2^*, \zeta), \end{cases}$$

where $\xi = x - \frac{1}{b+1}t^{b+1}$ and $\eta = x + \frac{1}{b+1}t^{b+1}$ are so-called characteristic coordinates used instead of φ^- and φ^+ respectively in this appendix and $z = 1 - \dfrac{\eta}{\xi}$, $\zeta = 1 - \dfrac{\xi}{\eta}$.

Now we define $U_\alpha^{(j)}(t,x)$ and $V_\alpha^{(j)}(t,x)$ as the solutions of the following Cauchy problems respectively:

For $j=0$, we set $U_\alpha^{(0)}(t,x) = U_\alpha(t,x)$ and $V_\alpha^{(0)}(t,x) = V_\alpha(t,x)$.

For $j \geq 1$, $\qquad P_c U_\alpha^{(j)} = t^{b-1} U_\alpha^{(j-1)}$

with the null initial data $\begin{cases} U_\alpha^{(j)}(0,x) = 0 \\ U_{\alpha t}^{(j)}(0,x) = 0 \end{cases}$, and $$P_c V_\alpha^{(j)} = t^{b-1} V_\alpha^{(j-1)}$$

with the null initial data $$\begin{cases} V_\alpha^{(j)}(0,x) = 0 \\ V_{\alpha\,t}^{(j)}(0,x) = 0 \end{cases}.$$

And then we define $X_\alpha(t,x)$, $Y_\alpha(t,x)$, $X_\alpha^{(j)}(t,x)$ and $Y_\alpha^{(j)}(t,x)$ as follows.

$$\begin{cases} X_\alpha(t,x) = \partial_\alpha U_\alpha(t,x) \quad \text{and } X_\alpha^{(j)}(t,x) = \partial_\alpha U_\alpha^{(j)}(t,x). \\ Y_\alpha(t,x) = \partial_\alpha V_\alpha(t,x) \quad \text{and } Y_\alpha^{(j)}(t,x) = \partial_\alpha V_\alpha^{(j)}(t,x). \end{cases}$$

Therefore X_α and Y_α satisfy the following Cauchy problem respectively.

$$\begin{cases} P_c X_\alpha = 0 \quad , \quad X_\alpha(0,x) = k_\alpha(x) \text{ and } X_{\alpha t}(0,x) = 0 \quad . \\ P_c Y_\alpha = 0 \quad , \quad Y_\alpha(0,x) = 0 \quad \text{and } Y_{\alpha t}(0,x) = k_\alpha(x) \quad . \end{cases}$$

$X_\alpha^{(j)}$ and $Y_\alpha^{(j)}$ satisfy the following Cauchy problem respectively for $j \geqq 1$.

$$\begin{cases} P_c X_\alpha^{(j)} = t^{b-1} X_\alpha^{(j-1)} \;,\; X_\alpha^{(j-1)}(0,x) = X_{\alpha\,t}^{(j-1)}(0,x) = 0 \quad . \\ P_c Y_\alpha^{(j)} = t^{b-1} Y_\alpha^{(j-1)} \;,\; Y_\alpha^{(j-1)}(0,x) = Y_{\alpha\,t}^{(j-1)}(0,x) = 0 \quad . \end{cases}$$

Remark that these auxiliary functions depend on c analytically but we omit c for brevity. For example we write $U_\alpha(t,x)$ instead of $U_\alpha(t,x,c)$ in this appendix.

The explicit representations of U_α and V_α lead to the explicit representations of $U_\alpha^{(j)}$, $V_\alpha^{(j)}$, $X_\alpha^{(j)}$ and $Y_\alpha^{(j)}$, as follows.

Lemma A.1

$$U_\alpha^{(j)}(t,x)=\frac{1}{j!}\partial_c^j U_{\alpha+j}(t,x)=\frac{1}{j!}\partial_c^j\left\{\frac{\xi^\alpha}{\Gamma(\alpha+j+1)}F(-\alpha-j,b^*+c^*,2b^*,z)\right\},$$

$$V_\alpha^{(j)}(t,x)=\frac{1}{j!}\partial_c^j V_{\alpha+j}(t,x)=\frac{1}{j!}\partial_c^j\left\{\frac{\xi^\alpha\ t}{\Gamma(\alpha+j+1)}F(-\alpha-j,c^*-b^*+1,1+2^*,z)\right\},$$

$$X_\alpha^{(j)}(t,x)=\frac{1}{j!}\partial_c^j X_{\alpha+j}(t,x)=\frac{1}{j!}\partial_c^j\,\partial_\alpha X_{\alpha+j}(t,x)$$
$$=\frac{1}{j!}\partial_c^j\partial_\alpha\left\{\frac{\xi^\alpha}{\Gamma(\alpha+j+1)}F(-\alpha-j,b^*+c^*,2b^*;z)\right\},$$

$$Y_\alpha^{(j)}(t,x)=\frac{1}{j!}\partial_c^j Y_{\alpha+j}(t,x)=\frac{1}{j!}\partial_c^j\,\partial_\alpha Y_{\alpha+j}(t,x)$$
$$=\frac{1}{j!}\partial_c^j\partial_\alpha\left\{\frac{\xi^\alpha\ t}{\Gamma(\alpha+j+1)}F(-\alpha-j,c^*-b^*+1,1+2^*,z)\right\}$$

LemmaA.1 shows that $U_\alpha^{(j)}$, $V_\alpha^{(j)}$, $X_\alpha^{(j)}$ and $Y_\alpha^{(j)}$ being represented as the derivatives by α and c of U_α and V_α, first we must study the auxiliary functions U_α and V_α .

Since U_α and V_α have hypergeometric functions as important factors respectively, the study of the multi-valuedness and the singularities of these auxiliary functions are reduced to those of hypergeometric functions. The next lemma about the analytic continuation, that is the monodromy theory of hypergeometric functions, plays a fundamental role in the study of the behaviour of these auxiliary functions around the branching surfaces.

Lemma A.2 Let S be the Riemann sphere. Put $D=S\setminus\{0,1,\infty\}$. Set $F=F(A,B,C,z)$ and $\widetilde{F}=z^{1-C}F(A-C+1,B-C+1,2-C,z)$ which constitute a fundamental system of solutions for the hypergeometric ordinary differential equation $\left[z(1-z)\frac{d^2}{dz^2}+\{C-(A+B+1)z\}\frac{d}{dz}-AB\right]u(z)=0$. Now. the monodromy representation ρ of the hypergeometric differential ordinary equation with respect to the fundamental system $\{F,\widetilde{F}\}$ is defined in the following way:

Denote by l_0 (l_1, l_∞ respectively) a loop which encircles the point 0 (1, ∞ respectively) once in the positive sense . We denote by the same letter l_0 (l_1, l_∞ respectively) a homotopy class containing l_0 (l_1, l_∞ respectively). Let $\pi = \pi(D)$ be the fundamental group of D. Then we can define a homomorphism ρ of the group π onto the group $G \subset GL(2,C)$ which is called the monodromy representation of the hypergeometric differential equation with respect to the fundamental system $\{F, \tilde{F}\}$, where G is the sub-group of GL(2,C), generated by g_0 and g_1 ,

$$\rho(l_0) = g_0 = \begin{pmatrix} 1 & , & 0 \\ 0 & , & e^{-2\pi i C} \end{pmatrix}$$

$$\rho(l_\infty) = g_\infty = (g_0 g_1)^{-1}$$

$$\rho(l_1) = g_1 = D^{-1}C^{-1} \begin{pmatrix} 1 & , & 0 \\ 0 & , & e^{2\pi i(C-B-A)} \end{pmatrix} CD$$

where

$$C = \begin{pmatrix} e^{-2\pi i A} - e^{-2\pi i C} & , & e^{-2\pi i(C-B)}(e^{-2\pi i A} - 1) \\ e^{-2\pi i B} - 1 & , & 1 - e^{-2\pi i(C-B)} \end{pmatrix}$$

$$D = \begin{pmatrix} \dfrac{\Gamma(B)\Gamma(C-B)}{\Gamma(C)} & , & 0 \\ 0 & , & \dfrac{\Gamma(A-C+1)\Gamma(1-A)}{\Gamma(2-C)} e^{\pi i(C+A-B-1)} \end{pmatrix}$$

$$C^{-1} = \frac{1}{|C|}\begin{pmatrix} 1 - e^{-2\pi i(C-B)} & , & 1 - e^{-2\pi iB} \\ e^{-2\pi i(C-B)}(1 - e^{-2\pi iA}) & , & e^{-2\pi iA} - e^{-2\pi iC} \end{pmatrix}$$

$$|C| = e^{-2\pi iA}(1 - e^{-2\pi iC})(1 - e^{-2\pi i(C-B-A)}) \quad .$$

Namely , if $\{F, \widetilde{F}\}$ is continued analytically along l_0 (l_1 , l_∞ respectively) ,then

$$\begin{pmatrix} F \\ \widetilde{F} \end{pmatrix} \text{ goes to } g_0\begin{pmatrix} F \\ \widetilde{F} \end{pmatrix} \left(g_1\begin{pmatrix} F \\ \widetilde{F} \end{pmatrix} , g_\infty\begin{pmatrix} F \\ \widetilde{F} \end{pmatrix} \text{ respectively} \right) .$$

(Correction: In Lemma 2.1 of the author's preceeding paper [10] D of the above lemma was dropped .)

We apply this lemma to the study of U_α and V_α . In our case , F is $F(\alpha;\xi,\eta)=F(-\alpha,b^*+c^*,2b^*,z)$ and so $\widetilde{F}$ is $\widetilde{F}(\alpha;\xi,\eta)= z^{1-2b^*}F(-\alpha-2b^*+1,c^*-b^*+1,2-2b^*,z)$. To explain the multi-valuedness of U_α and V_α in termes of the monodromy theory of hypergeometric functions , we introduce the new functions $\widetilde{U}_\alpha(t,x)$ and $\widetilde{V}_\alpha(t,x)$:

$$\widetilde{U}_\alpha(t,x) = \frac{\xi^\alpha}{\Gamma(\alpha+1)}\widetilde{F}(\alpha;\xi,\eta)$$

$$= \frac{\xi^{\alpha}}{\Gamma(\alpha+1)} z^{2^*} F(-\alpha+2^*, c^*-b^*+1, 1+2^*, z)$$

$$\tilde{V}_{\alpha}(t,x) = \frac{\xi^{\alpha} t}{\Gamma(\alpha+1)} z^{-2^*} F(-\alpha-2^*, c^*+b^*, 2b^*, z) \qquad .$$

Let $D_r = \{ (\xi,\eta);\ \xi \neq 0,\ \eta \neq 0,\ \xi \neq \eta\ , |\xi| < r,\ |\eta| < r \}$. Let P be any point belonging to D_r and keep P fixed. We consider any loop l starting and terminating at P in the domain D_r . Especially we denote by l(P) the terminating point in order to distinguish the terminating point from the starting point P . We denote by the same letter l the homotopy class containing l , too. Let $\pi = \pi(D_r, P)$ be the fundamental group of D_r with the base point P. Denote by $l_{\infty} \in \pi$ ($l_1 \in \pi$, $l_0 \in \pi$ respectively) a loop which encircles $\xi = 0$ ($\eta = 0, \xi = \eta$ respectively) once in the positive sense , where by " in the positive sense " we mean that the loop which is transformed by the mapping $z = 1 - \frac{\eta}{\xi}$ encircles the corresponding point in the positive sense in z-plane. We note that this mapping $z = 1 - \frac{\eta}{\xi}$ transforms $\xi = 0$, $\eta = 0, \xi = \eta$ to $z = \infty$, 1 , 0 respectively and l_0, l_1, l_{∞} to the corresponding loops in Lemma A.2

Using Lemma A.2 , we have

$$\begin{pmatrix} F(\alpha;l_0(P)) \\ \widetilde{F}(\alpha;l_0(P)) \end{pmatrix} = g_0 \begin{pmatrix} F(\alpha;P) \\ \widetilde{F}(\alpha;P) \end{pmatrix} , \quad \begin{pmatrix} F(\alpha;l_1(P)) \\ \widetilde{F}(\alpha;l_1(P)) \end{pmatrix} = g_1 \begin{pmatrix} F(\alpha;P) \\ \widetilde{F}(\alpha;P) \end{pmatrix} ,$$

$$\begin{pmatrix} F(\alpha;l_\infty(P)) \\ \widetilde{F}(\alpha;l_\infty(P)) \end{pmatrix} = g_\infty \begin{pmatrix} F(\alpha;P) \\ \widetilde{F}(\alpha;P) \end{pmatrix} .$$

Generally we have

$$\begin{pmatrix} F(\alpha;l(P)) \\ \widetilde{F}(\alpha;l(P)) \end{pmatrix} = \rho(l) \begin{pmatrix} F(\alpha;P) \\ \widetilde{F}(\alpha;P) \end{pmatrix} .$$

Therefore we have

$$\begin{pmatrix} U_\alpha(l_0(P)) \\ \widetilde{U}_\alpha(l_0(P)) \end{pmatrix} = g_0 \begin{pmatrix} U_\alpha(P) \\ \widetilde{U}_\alpha(P) \end{pmatrix}$$

$$\begin{pmatrix} U_\alpha(l_1(P)) \\ \widetilde{U}_\alpha(l_1(P)) \end{pmatrix} = g_1 \begin{pmatrix} U_\alpha(P) \\ \widetilde{U}_\alpha(P) \end{pmatrix}$$

$$\begin{pmatrix} U_\alpha(l_\infty(P)) \\ \widetilde{U}_\alpha(l_\infty(P)) \end{pmatrix} = e^{2\pi i\alpha} g_\infty \begin{pmatrix} U_\alpha(P) \\ \widetilde{U}_\alpha(P) \end{pmatrix}$$

We introduce the new monodromy representation ρ^* with respect to the pair $\{U_\alpha , \widetilde{U}_\alpha\}$ as a homomorphism ρ^* of the group π

onto the group $G^* \subset GL(2,C)$, where G^* is the sub-group of $GL(2,C)$ generated by g_0 , g_1 and $g^*_\infty = e^{2\pi i \alpha} g_\infty$;

$$\rho^*(1_0) = g_0 \quad , \quad \rho^*(1_1) = g_1 \quad , \quad \rho^*(1_\infty) = g^*_\infty \quad .$$

Then generally we have

$$\begin{pmatrix} U_\alpha(1(P)) \\ \widetilde{U}_\alpha(1(P)) \end{pmatrix} = \rho^*(1) \begin{pmatrix} U_\alpha(P) \\ \widetilde{U}_\alpha(P) \end{pmatrix}$$

Remark: we need only $U_\alpha(1(P))$, but we do not need $\widetilde{U}_\alpha(1(P))$. For the study of the multi-valuedness of $U_\alpha(1(P))$, we have to study $\rho^*(1)$ namely g_0 , g_1 , g^*_∞ , C and D . Seeing that both g_0 and D are diagonal matrices , they are commutative each other, and so $g_0 = D^{-1}g_0D$. on the other hand g_1 and g^*_∞ are also in the form $D^{-1}GD$, that is, $g_1 = D^{-1}E_1D$ and $g^*_\infty = D^{-1}E_\infty D$. Therefore, $\rho^*(1) = D^{-1}\rho^\#(1)D$ holds where $\rho^\#(1)$ is a homomorphism π_1 onto $G^\# \subset GL(2,C)$, the sub-group of $GL(2,C)$ generated by g_0 , E_1 and E_∞ . $\rho^\#(1)$, the principal part of $\rho^*(1)$, is to be investigated precisely.

$E_\infty = E_1^{-1} g_0^{-1}$ and E_1 generating $G^{\#}$ with the simple matrix g_0 ,

we have only to examine E_1 and E_1^{-1} .

Put $C = \begin{pmatrix} c_{11} , & c_{12} \\ c_{21} , & c_{22} \end{pmatrix}$ and then we have $C = \frac{1}{|C|}\begin{pmatrix} c_{22} & ,-c_{12} \\ -c_{21}, & c_{11} \end{pmatrix}$

Put $G = C^{-1}\begin{pmatrix} 1 & , 0 \\ 0 & , k \end{pmatrix} C$ .where $G = E_1$ or E_1^{-1} , and then we have

$$G = \frac{1}{|C|}\begin{pmatrix} c_{11}c_{22} - kc_{21}c_{12} & , \ (1-k)c_{12}c_{22} \\ (k-1)c_{11}c_{21} & , \ kc_{11}c_{22} - c_{21}c_{12} \end{pmatrix}$$

$$= \frac{1}{|C|}\begin{pmatrix} |C| + (1-k)c_{12}c_{21} & , \ (1-k)c_{12}c_{22} \\ -(1-k)c_{11}c_{21} & , \ |C| - (1-k)c_{11}c_{22} \end{pmatrix}$$

where $k = \begin{cases} e^{2\pi i(C-B-A)} & \text{in the case } G = E_1 \\ e^{-2\pi i(C-B-A)} & \text{in the case } G = E_1^{-1} \end{cases}$

So, it is possible that G has the singularity on

$$|C| = e^{-2\pi iA}(1-e^{-2\pi iC})(1-e^{-2\pi i(C-B-A)}) = 0 \quad .$$

We choose b such that Re $b > 0$ and keep fixed , so that we have $1-e^{-2\pi iC} \neq 0$. There is possibility that G has the singularities on $1 - e^{-2\pi i(C-B-A)} = 0$.

$$1 - e^{-2\pi i(C-B-A)} = \begin{cases} \frac{1}{k}(1-k) & \text{in the case } G = E_1 \\ (1-k) & \text{in the case } G = E_1^{-1} \end{cases} .$$

Put $G = \frac{1}{|C|}\begin{pmatrix} G_{11} , & G_{12} \\ G_{21} , & G_{22} \end{pmatrix}$ and then from the above , every component G_{ij} of G has the factor $1-k$. Therefore, the singularity on $1 - e^{-2\pi i(C-B-A)} = 0$ of G is removable . So we see E_1 and E_1^{-1} are entire functions of c and α .

Lemma A.3 $\rho^{\#}(l)$ is a entire function of c and α .

$\rho^{\#}(l)$ is a periodic function of α with period 1 and $\rho^{\#}(l)$ is a periodic function of c with period $2(b+1)$, too .

Put $\rho^{\#}(l) = \begin{pmatrix} \rho^{\#}_{11} , & \rho^{\#}_{12} \\ \rho^{\#}_{21} & \rho^{\#}_{22} \end{pmatrix}$ and $D = \begin{pmatrix} d_1 , & 0 \\ 0 , & d_2 \end{pmatrix}$

Then we have

$$\rho^{*}(l) = \begin{pmatrix} \rho^{\#}_{11} , & d_1^{-1}d_2\, \rho^{\#}_{12} \\ d_1 d_2^{-1} \rho^{\#}_{21} , & \rho^{\#}_{22} \end{pmatrix} .$$

By $\begin{pmatrix} U_\alpha(l(P)) \\ \widetilde{U}_\alpha(l(P)) \end{pmatrix} = \rho^{*}(l) \begin{pmatrix} U_\alpha(P) \\ U_\alpha(P) \end{pmatrix}$, we have reached the next formula which links U_α on the universal covering space with U_α and $\widetilde{U}_\alpha$ on the base space.

(F.A) $U_\alpha(l(P)) = \rho^{\#}_{11}(l)U_\alpha(P) +$

$$\frac{\Gamma(C)\Gamma(A-C+1)\Gamma(1-A)}{\Gamma(B)\Gamma(C-B)\Gamma(2-C)}e^{\pi i(C+A-B-1)}\rho^{\#}_{12}(l)\widetilde{U}_\alpha(P) \quad .$$

Note (1) Putting $A=-\alpha$, $B=c^*-b^*+1$, $C=1+2^*$, instead of $A=-\alpha, B=b^*+c^*$, $C=2b^*$, we have the formula of $V_\alpha(l(P))$.

Note (2) In this formula , Γ-factor of the right hand side has no pole if α is a positive integer .

Secondly after expressing U_α (V_α) on the universal covering space with U_α and $\hat{U}_\alpha$ (V_α and $\widetilde{V}_\alpha$) on the base space we invetigate U_α and $\widetilde{U}_\alpha$ (V_α and $\widetilde{V}_\alpha$) on the base space with means of the connection formulae of the hypergeometric functions which is described below .

(C.F.) (1) $F(A,B,C,z) = \frac{\Gamma(A+B-C)\Gamma(C)}{\Gamma(A)\Gamma(B)}(1-z)^{C-A-B}F(C-A,C-B,C-A-B+1,$

$,1-z) + \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)}F(A,B,A+B-C+1,1-z)$ in $|1-z| = \left|\frac{\eta}{\xi}\right| < 1$.

(2) $F(A,B,C,z) = \frac{\Gamma(B-A)\Gamma(C)}{\Gamma(C-A)\Gamma(B)}(1-z)^{-A}F(A,C-B,A-B+1,\frac{1}{1-z})$

$$+\frac{\Gamma(A-B)\Gamma(C)}{\Gamma(C-B)\Gamma(A)}(1-z)^{-B}F(B,C-A,B-A+1.\frac{1}{1-z}) \quad \text{in } \left|\frac{1}{1-z}\right| = \left|\frac{\xi}{\eta}\right| < 1 \quad .$$

Using these connection formulae by the help of Kummer's transformations , we list up the representations of U_α , $\widetilde{U}_\alpha$, V_α and $\widetilde{V}_\alpha$ in $\left|\frac{\eta}{\xi}\right| \leqq 1$ and $\left|\frac{\xi}{\eta}\right| \leqq 1$ respectively. These following formulae (F.B) are fundamental formulae by which we describe the singularities of U_α, $U_\alpha^{(j)}$, V_α, $V_\alpha^{(j)}$, X_α, $X_\alpha^{(j)}$, Y_α and $Y_\alpha^{(j)}$.

(F.B) (U_α, $\widetilde{U}_\alpha$, V_α, $\widetilde{V}_\alpha$ on the base space)

$$\begin{aligned}
U_\alpha &= \gamma_1(\alpha,c)F_1(\alpha,c,\frac{\eta}{\xi})\frac{\eta^\alpha}{\Gamma(\alpha+1)}\left(\frac{\eta}{\xi}\right)^{b^*-c^*} \\
&\qquad + \gamma_2(\alpha,c)F_2(\alpha,c,\frac{\eta}{\xi})\frac{\xi^\alpha}{\Gamma(\alpha+1)} \qquad \text{in } \left|\frac{\eta}{\xi}\right| \leqq 1 \\
&= \gamma_2(\alpha,-c)F_2(\alpha,-c,\frac{\xi}{\eta})\frac{\eta^\alpha}{\Gamma(\alpha+1)} \\
&\qquad + \gamma_1(\alpha,-c)F_1(\alpha,-c,\frac{\xi}{\eta})\frac{\xi^\alpha}{\Gamma(\alpha+1)}\left(\frac{\xi}{\eta}\right)^{b^*+c^*} \quad \text{in } \left|\frac{\xi}{\eta}\right| \leqq 1 \\
\widetilde{U}_\alpha &= \gamma_3(\alpha,c)F_1(\alpha,c,\frac{\eta}{\xi})\frac{\eta^\alpha}{\Gamma(\alpha+1)}\left(\frac{\eta}{\xi}\right)^{b^*-c^*} \\
&\qquad + \gamma_4(\alpha,c)F_2(\alpha.c,\frac{\eta}{\xi})\frac{\xi^\alpha}{\Gamma(\alpha+1)} \qquad \text{in } \left|\frac{\eta}{\xi}\right| \leqq 1 \\
&= \gamma_4(\alpha,-c)F_2(\alpha,-c,\frac{\xi}{\eta})\frac{\eta^\alpha}{\Gamma(\alpha+1)} \\
&\qquad + \gamma_3(\alpha,-c)F_1(\alpha,-c,\frac{\xi}{\eta})\frac{\xi^\alpha}{\Gamma(\alpha+1)}\left(\frac{\xi}{\eta}\right)^{b^*+c^*} \quad \text{in } \left|\frac{\xi}{\eta}\right| \leqq 1
\end{aligned}$$

$$V_\alpha = \gamma_3(\alpha+2^*,c)F_1(\alpha+2^*,c,\frac{\eta}{\xi})\frac{\eta^{\alpha+2^*}}{\Gamma(\alpha+1)}\left(\frac{\eta}{\xi}\right)^{b^*-c^*}K$$

$$+\gamma_4(\alpha+2^*,c)F_2(\alpha+2^*,c,\frac{\eta}{\xi})\frac{\xi^{\alpha+2^*}}{\Gamma(\alpha+1)}K \qquad \text{in } \left|\frac{\eta}{\xi}\right|\leq 1$$

$$=\gamma_4(\alpha+2^*.-c)F_2(\alpha+2^*.-c,\frac{\xi}{\eta})\frac{\eta^{\alpha+2^*}}{\Gamma(\alpha+1)}K$$

$$+\gamma_3(\alpha+2^*.-c)F_1(\alpha+2^*,-c,\frac{\xi}{\eta})\frac{\xi^{\alpha+2^*}}{\Gamma(\alpha+1)}\left(\frac{\xi}{\eta}\right)^{b^*+c^*}K \qquad \text{in } \left|\frac{\xi}{\eta}\right|\leq 1$$

$$\widetilde{V}_\alpha = \gamma_1(\alpha+2^*.c)F_1(\alpha+2^*,c,\frac{\eta}{\xi})\frac{\eta^{\alpha+2^*}}{\Gamma(\alpha+1)}\left(\frac{\eta}{\xi}\right)^{b^*-c^*}K$$

$$+\gamma_2(\alpha+2^*.c)F_2(\alpha+2^*,c,\frac{\eta}{\xi})\frac{\xi^{\alpha+2^*}}{\Gamma(\alpha+1)}K \qquad \text{in } \left|\frac{\eta}{\xi}\right|\leq 1$$

$$=\gamma_2(\alpha+2^*,-c)F_2(\alpha+2^*,-c,\frac{\xi}{\eta})\frac{\eta^{\alpha+2^*}}{\Gamma(\alpha+1)}K$$

$$+\gamma_1(\alpha+2^*,-c)F_1(\alpha+2^*.-c,\frac{\xi}{\eta})\frac{\xi^{\alpha+2^*}}{\Gamma(\alpha+1)}\left(\frac{\xi}{\eta}\right)^{b^*+c^*}K \qquad \text{in } \left|\frac{\xi}{\eta}\right|\leq 1$$

,where $\gamma_1(\alpha,c) = \dfrac{\Gamma(-\alpha-b^*+c^*)\Gamma(2b^*)}{\Gamma(-\alpha)\Gamma(b^*+c^*)}$,

$\gamma_2(\alpha,c) = \dfrac{\Gamma(\alpha+b^*-c^*)\Gamma(2b^*)}{\Gamma(\alpha+2b^*)\Gamma(b^*-c^*)}$,

$\gamma_3(\alpha,c) = \dfrac{\Gamma(-\alpha-b^*+c^*)\Gamma(1+2^*)}{\Gamma(-\alpha+2^*)\Gamma(c^*-b^*+1)}$,

$\gamma_4(\alpha,c) = \dfrac{\Gamma(\alpha+b^*-c^*)\Gamma(1+2^*)}{\Gamma(\alpha+1)\Gamma(b^*-c^*+2^*)}$,

$F_1(\alpha.c,z) = F(\alpha+2b^*,b^*-c^*.\alpha+1+b^*-c^*,z)$,

$F_2(\alpha,c,z) = F(-\alpha,b^*+c^*,-\alpha+c^*-b^*+1,z)$,

and $K = (-4^*)^{-\frac{1}{b+1}}$.

Thus we have seen the multivaluedness of U_α and V_α .

For the proof of the convergence of the formal solution, we need the estimates of U_α and V_α on the universal covering space, which are reduced to the estimates of U_α and V_α on the base space . By (F.B), those are reduced to the estimates of the hypergeometric series $F_1(\alpha,c,z)$ and $F_2(\alpha,c,z)$, which are based on the following lemmata. Let b be a real positive number and keep fixed henceforth.

Lemma A.4 $F_1(\alpha,c,z)=F(\alpha+2b^*,b^*-c^*,\alpha+b^*-c^*+1,z)$

$$KF(\alpha+2b^*,b^*+|\mathrm{Re}\ c^*|+\varepsilon|\mathrm{Im}\ c^*|,\alpha+1+b^*+|\mathrm{Re}\ c^*|+\varepsilon|\mathrm{Im}\ c^*|,z)$$

for $|z|\leqq 1$, $\alpha\geqq 0$, $\varepsilon\geqq 1$,

where K is a constant depending only on c .

Lemma A.5 $F_2(\alpha,c,z)=F(-\alpha,b^*+c^*,-\alpha-b^*+c^*+1,z)$

$$T^{\alpha+[|\mathrm{Re}\ c^*|]+1}F(\alpha,b^*+|\mathrm{Re}\ c^*|+\varepsilon|\mathrm{Im} c^*|,\alpha+1-b^*+|\mathrm{Re}\ c^*|+\varepsilon|\mathrm{Im}\ c^*|,z)$$

for $\varepsilon\geqq 0$, $\alpha>0$,

where T is a constant depending only on c and b , and independent of α .

Remark: the proof of Lemma A.4 is easy, but the proof of Lemma A.5 is done with only tedious calculations. So we omit them. The result about the exceptional values of parameters of the hypergeometric series are to be described later

Thirdly we are to study $U_{\alpha}^{(j)}$, $V_{\alpha}^{(j)}$, $X_{\alpha}^{(j)}$ and $Y_{\alpha}^{(j)}$ By Lemma A.1, we need to diferentiate U_{α} and V_{α} by c several times and by α at most once. By (F.A), the differentiations of U_{α} and V_{α} are reduced mainly to the differentiations of the hypergeometric series. And so we need the following lemma about the differentiation of the product of Γ-functions.

Lemma A.6 Let H(c) be an infinitely differentiable function and put $\frac{d}{dc}H(c)=H'(c)=H(c)\psi(c)$. Then we have the next formula about $(\frac{d}{dc})^n H(c)$

$$(\frac{d}{dc})^n H(c)=H^{(n)}(c)=\sum_{(s_1,\cdots,s_n)}\frac{n!}{s_1!\cdots s_n!}(\frac{\psi}{1!})^{s_1}(\frac{\psi'}{2!})^{s_2}\cdots(\frac{\psi^{(n)}}{n!})^{s_n}$$

where $(s_1,\cdots,s_n)$ varies in P_n, that is, the set of all $(s_1,\cdots,s_n)$ such that $\sum_{k=1}^{n} ks_k = n$ and $s_1\geqq 0,\cdots,s_n\geqq 0$. (See [8].)

We explain how to use this lemma. For example, in the case $H(c)=\frac{\Gamma(c+p)}{\Gamma(c+q)}$, we put $H'(c)=\frac{\Gamma(c+p)}{\Gamma(c+q)}\left(\frac{\Gamma'(c+p)}{\Gamma(c+p)}-\frac{\Gamma'(c+q)}{\Gamma(c+q)}\right)=H(c)\psi(c)$

Namely $\psi(c)=\psi(c+p)-\psi(c+q)$, where ψ of the right hand side of this identity is di-Γfunction. Taking account of the fact that the formula about the poly-Γfunction $\psi^{(n)}(z+1)=\psi^{(n)}(z)+(-1)^n\frac{n!}{z^{n+1}}$ holds except the nonpositive integer z, we are able to estimate $\psi^{(n)}(c+p)-\psi^{(n)}(c+q)$ for $n\geq 1$

The next lemma about the cardinal number of P_n, which we denote by $p(n)$, are to be used for the estimation of derivatives of F_1 and F_2

Lemma A.7 $p(n)$ is called the partition number of n The following estimates of $p(n)$ are known.

$$Hn^{-1}\ e^{2\sqrt{n}} < p(n) < Kn^{-1}\ e^{2\sqrt{2n}}$$

, where H and K are absolute constants. (See [4].)

To see $X_\alpha^{(j)}(P)$ and $Y_\alpha^{(j)}(P)$ on the base space, we have only to see $\partial_c^l\partial_\alpha F_1(\alpha,c,z)$ and $\partial_c^l\partial_\alpha F_2(\alpha,c,z)$ in $|z|\leq 1$ We put

$$F_1(\alpha,c,z)=\sum_{n=0}^{+\infty}\frac{\Gamma(\alpha+2b^*+n)\Gamma(b^*-c^*+n)\Gamma(\alpha+b^*-c^*+1)}{\Gamma(\alpha+2b^*)\Gamma(b^*-c^*)\Gamma(\alpha+b^*-c^*+1+n)}\frac{z^n}{n!}=\sum_{n=0}^{+\infty}H_n(c)\frac{z^n}{n!}\ .$$

$$\partial_\alpha F_1=\sum_{n=0}^{+\infty}H_n(c)\left[\sum_{r=0}^{n-1}\left\{\frac{1}{\alpha+2b^*+r}-\frac{1}{\alpha+b^*-c^*+1+r}\right\}\right]\frac{z^n}{n!}\ ,$$

$$\partial_c\partial_\alpha F_1=\sum_{n=0}^{+\infty}\Bigg[H_n(c)\left(\sum_{r=0}^{n-1}\left\{\frac{1}{b^*-c^*+r}-\frac{1}{\alpha+b^*-c^*+1+r}\right\}\right)\left(\sum_{r=0}^{n}\left\{\frac{1}{\alpha+2b^*+r}-\frac{1}{+b^*-c^*+1+r}\right\}\right)+H_n(c)\sum_{r=0}^{n-1}\frac{1}{(\alpha+b^*-c^*+1+r)^2}\Bigg]\frac{1}{2(b+1)}\frac{z^n}{n!}\ .$$

Differentiating by c this identity several times and applying Lemma A.6 to the coefficients of these power series, we are to see $\partial_c^l\partial_\alpha F_1$ and $\partial_c^l\partial_\alpha F_2$ inductively and to estimate them by the help of Lemma A.7. In fact, for example, put $\psi_n(c)=\sum_{r=0}^{n-1}\left\{\frac{1}{b^*-c^*+r}-\frac{1}{\alpha+b^*-c^*+1+r}\right\}$ and then we have $|\psi_n(c)|<C$ and $|\partial_c^l\psi_n(c)|<l!C^l$, where α is a non-negative integer. As for $\partial_c^l\partial_\alpha F_2$, we can treat it in the similar way. By the above lemmata and the formulae with respect to poly-Γfunctions, we have

$$\partial_c^l\partial_\alpha F_1\ll Bl!C^lKF(\alpha+2b^*,b^*+|\mathrm{Re}\ c^*|+\varepsilon|\mathrm{Im}\ c^*|,\alpha+1+b^*+|\mathrm{Re}\ c^*|+\varepsilon|\mathrm{Im}\ c^*|,z)\quad \text{for } \alpha\geqq 0,\ \varepsilon\geqq 1\ ,\quad\text{and}$$

$$\partial_c^l\partial_\alpha F_2\ll Bl!C^lK^{+[|\mathrm{Re}\ c^*|]+1}F(\alpha,b^*+|\mathrm{Re}\ c^*|+\varepsilon|\mathrm{Im}\ c^*|,$$

$\alpha+1-b^*+|\mathrm{Re}\ c^*|+\varepsilon|\mathrm{Im}\ c^*|, z)$ for $\alpha>0$, $\varepsilon \geqq 0$,

where B,C and K are suitable constants independent of α and l .

To estimate the hypergeometric series in the convergent circle , we need the following lemma.

Lemmma A.8 If A,B,C and C-B-A are positive, we have

$$F(A,B,C,z) = F(A,B,C,1) = \frac{\Gamma(C)\Gamma(C-B-A)}{\Gamma(C-A)\Gamma(C-B)} \quad \text{in } |z| \leqq 1$$

,(see [10]) .

Therefore we have the following estimates of $\partial_c^l \partial_\alpha F_1$ and $\partial_c^l \partial_\alpha F_2$ in $|z| \leqq 1$.

$$\left|\partial_c^l \partial_\alpha F_1(\alpha,c,z)\right| = Bl!C^l K \left|\frac{\Gamma(\alpha+1+b^*+|\mathrm{Re}\ c^*|+\varepsilon|\mathrm{Im}\ c^*|)\Gamma(1-2b^*)}{\Gamma(\alpha+1)\Gamma(1-b^*+|\mathrm{Re}\ c^*|+\varepsilon|\mathrm{Im}\ c^*|)}\right| \text{ in } |z| \leqq 1,$$

$$\left|\partial_c^l \partial_\alpha F_2(\alpha,c,z)\right| = Bl!C^l K^{\alpha+[|\mathrm{Re}\ c^*|]+1}$$

$$\times \frac{\Gamma(\alpha+1-b^*+|\mathrm{Re}\ c^*|+\varepsilon|\mathrm{Im}\ c^*|)\Gamma(1-2b^*)}{\Gamma(\alpha+1-2b^*)\Gamma(1-b^*+|\mathrm{Re}\ c^*|+\varepsilon|\mathrm{Im}\ c^*|)} \text{ in } |z| \leqq 1,$$

for $\alpha>0$ and $\varepsilon \geqq 1$.

These estimates lead to the estimates of $X_\alpha^{(j)}(P)$, $Y_\alpha^{(j)}(P)$, and then it remains to estimate $X_\alpha^{(j)}$ and $Y_\alpha^{(j)}$ on the universal covering space. With respect to $\Gamma^{\#} = \dfrac{\Gamma(C)\Gamma(A-C+1)\Gamma(1-A)}{\Gamma(B)\Gamma(C-B)\Gamma(2-C)}$ in

(F.A) , we rewrite as follows;

$$\Gamma^{\#} = \begin{cases} \gamma_5(\alpha, c) \equiv \dfrac{\Gamma(2b^*)\Gamma(-\alpha-2b^*+1)\Gamma(1+\alpha)}{\Gamma(b^*+c^*)\Gamma(b^*-c^*)\Gamma(2-2b^*)} & \text{in the case } U_\alpha(l(p)), \\ \gamma_6(\alpha, c) \equiv \dfrac{\Gamma(2-2b^*)\Gamma(-\alpha-1+2b^*)\Gamma(1+\alpha)}{\Gamma(1-b^*+c^*)\Gamma(1-b^*-c^*)\Gamma(2b^*)} & \text{in the case } V_\alpha(l(P)). \end{cases}$$

(Note: γ_5 and γ_6 have the meaning in the case b^*+c^* or b^*-c^* is an integer.)

Applying Lemma A.6 and A.7 to the estimations of $\gamma_i(\alpha, c)$ (i=1, ...,6) , we have

$$\left|\partial_c^l \partial_\alpha \gamma_1(\alpha, c)\right| \leqq Bl!C^l \left|\frac{\Gamma(1+\alpha)\Gamma(2b^*)}{\Gamma(1+\alpha+b^*-c^*)\Gamma(b^*+c^*)}\right| \left|\psi(1+\alpha+b^*+|c^*|)\right| ,$$

$$\left|\partial_c^l \partial_\alpha \gamma_2(\alpha, c)\right| \leqq Bl!C^l \left|\gamma_2(\alpha, c)\right| \left|\psi(\alpha+b^*+|c^*|)\right| ,$$

$$\left|\partial_c^l \partial_\alpha \gamma_3(\alpha, c)\right| \leqq Bl!C^l \left|\frac{\Gamma(\alpha+2b^*)\Gamma(2-2b^*)}{\Gamma(\alpha+1+b^*-c^*)\Gamma(1+c^*-b^*)}\right| \left|\psi(1+\alpha+b^*+|c^*|)\right| ,$$

$$\left|\partial_c^l \partial_\alpha \gamma_4(\alpha, c)\right| \leqq Bl!C^l \left|\gamma_4(\alpha, c)\right| \left|\psi(\alpha+b^*+|c^*|)\right| ,$$

$$\left|\partial_c^l \partial_\alpha \gamma_5(\alpha, c)\right| \leqq Bl!C^l \left|\gamma_5(\alpha, c)\right| ,$$

$$\left|\partial_c^l \partial_\alpha \gamma_6(\alpha, c)\right| \leqq Bl!C^l \left|\gamma_6(\alpha, c)\right| .$$

Using these estimates, we obtain the following estimates of $\partial_c^l \partial_\alpha U_\alpha$, $\partial_c^l \partial_\alpha U_\alpha$, $\partial_c^l \partial_\alpha V_\alpha$ and $\partial_c^l \partial_\alpha V_\alpha$ on the base space.

$$\left|\partial_c^l \partial_\alpha U_\alpha(P)\right| \leqq K^{\alpha+[|\mathrm{Re}\ c^*|]+1} BC^{l+2} l^4\ l! \frac{\psi(\alpha+1+b^*+|c^*|)}{\Gamma(1+\alpha)} \times$$

$$\operatorname{Max}(|\gamma_1(\alpha.c)|, |\gamma_2(\alpha,c)|)\, r^{\alpha+1-(\mathrm{Re}\ c^*)_+}(\log r)^{l+1}$$

in $|\xi| \leq r$ and $|\eta| \leq r$,

$$\left|\partial_c^l \partial_\alpha \tilde{U}_\alpha(P)\right| \leqq K^{\alpha+|\mathrm{Re}\ c^*|+1} B C^{l+2} l^4\ l!\frac{\Psi(\alpha+1+b^*+|c^*|)}{\Gamma(1+\alpha)} \times$$

$$\operatorname{Max}(|\gamma_3(\alpha,c)|, |\gamma_4(\alpha,c)|)\, r^{\alpha+1+(\mathrm{Re}\ c^*)_+}(\log r)^{l+1}$$

in $|\xi| \leq r$ and $|\eta| \leq r$,

$$\left|\partial_c^l \partial_\alpha V_\alpha(P)\right| \leqq K^{\alpha+2^*+|\mathrm{Re}\ c^*|+1} B C^{l+2} l^4\ l!\frac{\Psi(\alpha+2^*+1+b^*+|c^*|)}{\Gamma(1+\alpha+2^*)} \times$$

$$\operatorname{Max}(|\gamma_1(\alpha+2^*,c)|, |\gamma_2(\alpha+2^*,c)|)\, r^{\alpha+2^*+1-(\mathrm{Re}\ c^*)_+}(\log r)^{l+1}$$

in $|\xi| \leqq r$ and $|\eta| \leqq r$,

$$\left|\partial_c^l \partial_\alpha \tilde{V}_\alpha(P)\right| = K^{\alpha+2^*+|\mathrm{Re}\ c^*|+1} B C^{l+2} l^4\ l!\frac{\Psi(\alpha+2^*+1+b^*+|c^*|)}{\Gamma(1+\alpha+2^*)} \times$$

$$\operatorname{Max}(|\gamma_3(\alpha+2^*,c)|, |\gamma_4(\alpha+2^*.c)|)\, r^{\alpha+2^*+1+(\mathrm{Re}\ c^*)_+}(\log r)^{l+1}$$

in $|\xi| \leqq r$ and $|\eta| \leqq r$,

where $(\mathrm{Re}\ A)_+ = \operatorname{Max}(0, A)$

Taking account of the fact that $\rho^{\#}_{ij}$ are entire functions of α and c and periodic functions of α with period 1 , we have reached , by the help of all estimates obtained above,

(E.X) $\quad \left|X_\alpha^{(j)}(l(P))\right| \leqq C(l(P))K^{\alpha+j}C^j\frac{\Psi(\alpha+j+b^*+|c^*|+1)}{\Gamma(\alpha+j+1)} \times$

$$\operatorname{Max}(|\gamma_1(\alpha+j,c)|,|\gamma_2(\alpha+j,c)|,|\gamma_5(\alpha+j,c)\gamma_3(\alpha+j,c)|,$$

$$|\gamma_5(\alpha+j,c)\gamma_4(\alpha+j,c)|)r^{\alpha+j-|\mathrm{Re}\ c^*|}(\log r)^{j+1}$$

for $|\xi|<r$ and $|\eta|<r$, and

(E.Y) $$\left|Y_\alpha^{(j)}(l(P))\right| \leqq C(l(P))K^{\alpha+j}C^j\frac{\Psi(\alpha+2^*+j+b^*+|c^*|+1)}{\Gamma(\alpha+j+1)}\times$$

$$\operatorname{Max}(|\gamma_1(\alpha+2^*+j,c)|,|\gamma_2(\alpha+2^*+j,c)|,|\gamma_5(\alpha+2^*+j,c)\gamma_3(\alpha+2^*+j,c)|,$$

$$|\gamma_5(\alpha+2^*+j,c)\gamma_4(\alpha+2^*+j,c)|)r^{\alpha+2^*+j-|\mathrm{Re}\ c^*|}(\log r)^{j+1}$$

for $|\xi|<r$ and $|\eta|<r$,

where $C(l(P))$ is a constant depending only on $l(P)$ and $\alpha+j$ is a non-negative integer.

Lastly we obtain the following estimates, by Stirling's formula and the other formulae with respect to Γ-function.

$$\left|X_\alpha^{(j)}\right|,\left|Y_\alpha^{(j)}\right| < C_K\frac{\Psi(\alpha+j+1)}{\Gamma(\alpha+j+1)}C^{\alpha+j}r^{\alpha+j}(\log r)^j$$

on any compact set K of the universal covering space over $D_r\setminus\{\xi=0,\eta=0.\xi=\eta\}$, where $\alpha+j$ is a positive integer and C_K is a constant which depends only on K , and $D_r=\{(t,x):|\xi|<r,|\eta|<r\}$, and C is a constant independent of α, j and r .

Remark:

The above treatment and estimates of hypergeometric series, that is, U_α, V_α, $U_\alpha^{(j)}$, $V_\alpha^{(j)}$, $X_\alpha^{(j)}$, $Y_\alpha^{(j)}$ on the base space, are not valid in the case c is an exceptional value of parameters of hypergeometric series. To be precise, as to U_α, in two following cases,

Case I $c^*-b^* = m \in Z$ (in $\left|\frac{\eta}{\xi}\right| \leqq 1$)

Case II $c^*+b^* = n \in Z$ (in $\left|\frac{\xi}{\eta}\right| \leqq 1$)

$$\left(\begin{array}{lll} \text{Case I} \quad c^*-b^*-2^* = m \in Z & (\text{ in } \left|\frac{\eta}{\xi}\right| \leqq 1) & \\ & & \text{, as to } V_\alpha \\ \text{Case II} \quad c^*+b^*+2^* = n \in Z & (\text{ in } \left|\frac{\xi}{\eta}\right| \leqq 1) & \end{array}\right)$$

, (F.B), especially (C.F.) have troubles .

We explain how to overcome these difficulties, getting an example in Case II. Put $c^*+b^*=n$. We rewrite the following connection formula

$$F(-\alpha,n,2b^*,z)=\frac{\Gamma(n+\alpha)\Gamma(2b^*)}{\Gamma(2b^*+\alpha)\Gamma(n)}(1-z)^{\alpha}F(-\alpha,2b^*-n,-\alpha-n+1,\frac{1}{1-z}) +$$

$$\frac{\Gamma(-\alpha-n)\Gamma(2b^*)}{\Gamma(2b^*-n)\Gamma(-\alpha)}(1-z)^{-n}F(n,2b^*+\alpha,n+\alpha+1,\frac{1}{1-z}) \quad \text{in } \left|\frac{\xi}{\eta}\right| \leqq 1 ,$$

when n is positive , into the form,

$$\sin\pi(n+\alpha)F(-\alpha,n,2b^*,z)=\frac{\Gamma(2b^*)\Gamma(1+\alpha)\Gamma(1+n-2b^*)}{\Gamma(2b^*+\alpha)\Gamma(n)}(1-z)^{\alpha}F_1^*(z)$$

$$+\frac{\Gamma(2b^*)\Gamma(1-2b^*-\alpha)}{\Gamma(2b^*-n)\Gamma(n)\Gamma(-\alpha)}(1-z)^{-n}F_2^*(z) \qquad \text{in } \left|\frac{\xi}{\eta}\right|\leqq 1$$

, where $F_1^*(z)=\sum_{k=0}^{+\infty}(1-z)^{-k}\left[k!\Gamma(1+\alpha-k)\Gamma(1+n-2b^*-k)\Gamma(-\alpha-n+1+k)\right]^{-1}$

& $F_2^*(z)=\sum_{k=0}^{+\infty}(1-z)^{-k}\Gamma(n+k)\left[k!\Gamma(1-2b^*-\alpha-k)\Gamma(n+\alpha+k+1)\right]^{-1}$

and when n is non-positive, into the form

$$\frac{\sin\pi(n+\alpha)}{\pi}F(-\alpha,n,2b^*,z)=\frac{\Gamma(2b^*)\Gamma(1+\alpha)\Gamma(1+n-2b^*)}{\Gamma(2b^*+\alpha)\Gamma(n)}(1-z)^{\alpha}F_3^*(z)$$

$$+\frac{\Gamma(2b^*)\Gamma(1-2b^*-\alpha)\Gamma(1-n)}{\Gamma(2b^*-n)\Gamma(-\alpha)}(1-z)^{-n}F_4^*(z) \qquad \text{in } \left|\frac{\xi}{\eta}\right|\leqq 1$$

, where $F_3^*(z)=\sum_{k=0}^{+\infty}(1-z)^{-k}\left[k!\Gamma(1+\alpha-k)\Gamma(1+n-k-2b^*)\Gamma(1-\alpha-n+k)\right]^{-1}$

& $F_4^*(z)=\sum_{k=0}^{+\infty}(1-z)^{-k}\left[k!\Gamma(1-n-k)\Gamma(1-2b^*-\alpha-k)\Gamma(1+n+\alpha+k)\right]^{-1}$.

After operating ∂_c^{j+1} and $\partial_c^{j+1}\partial_\alpha$ to the both sides of this new connection formula, we set $b^*+c^*=n$ to be an integer , and we can know and estimate $\partial_\alpha\partial_c^jF(-\alpha,n,2b^*,z)$ in $\left|\frac{\xi}{\eta}\right|\leqq 1$ inductively. We can treat other cases in the similar way.

Remark:

The initial surface t=0 is corresponded to $\xi-\eta=0$. The characteristic roots of hypergeometric ordinary differential operator at z=0 are 0 and $1-C=\frac{1}{b+1}$. On the other hand, $\eta-\xi=\frac{2}{b+1}t^{b+1}$. Therefore the initial surface t=0 is not a branching surface.

We need the results in Appendix in the case b=1 for this paper, and in the case $b=\frac{1}{2}$ for the author's preceeding papers [10] and [11] .

We finish this paper with the following example.

(Example)

$$L=\partial_t^2-t^{2b}\partial_x^2-ct^{b-1}\partial_x-at^{b-1} \qquad (\text{ a is a constant })$$

We consider the following Cauchy problems with the singular data.

(1) $Lu(t,x)=0$

with the initial data $\begin{cases} u(0,x)=k_\alpha(x) \\ u_t(0,x)=0 \end{cases}$

$$u(t,x)=\sum_{k=0}^{+\infty}a^k X_\alpha^{(k)}(t,x) \qquad .$$

(2) $Lv(t,x) = 0$

with the initial data $\begin{cases} v(0,x) = 0 \\ v_t(0,x) = k_\alpha(x) \end{cases}$

$$v(t,x) = \sum_{k=0}^{+\infty} a^k Y_\alpha^{(k)}(t,x)$$

$u(t,x)$ ($v(t,x)$) is a unique holomorphic solution of the Cauchy problem (1) ((2)) on the universal covering space over $D_r \setminus \{\xi = 0, \eta = 0\}$

Depertment of Engineering,

Doshisha University.

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